MATHEMATICAL LOGIC

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PREFACE

This book is intended for honours and post graduate students in Mathematics and Philosophy of the Indian Universities. In the last chapter we have also set up the foundations of mathematics. Planned with a variety of needs in mind, we have attempted to make the exposition as elementary as possible.

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INTRODUCTION

1- **Logica Magna.** Logic goes back, as a systematic study to Aristotle, whose organon (instrument of reasoning] consists of six books and is an analogue of Euclid's geometry. Growing from the soil of mathematics, mathematical or symbolic logic has two aspects. On one hand it is logic—it is the science of the valid process of reasoning whose goal is to systematize and codify principles of valid reasoning by mathematical methods. The other aspect of mathematical logic is interlaced with setting up the foundations of mathematics, which amounts to the formalization of a mathematical theory as a logical system augmented by non-logical axioms. But it is also possible to develop a logical system which leads us to deduce the totality of pure mathematics without an appeal to any specific axioms and is called Logica Magna [or a Grand Logic]. Principia Mathematica and its revision by Carnap, Church, Tarki, the system of S. Les'niewki, the systems of Quine's New foundations, Skolem's formalisations of the systems of set theory, Heyting's intuitionitic logic and others......may be considered as a Logica Magna.

2. **Statements.** As Logic is regarded as an applied mathematics, we must choose its appropriate subject matter. It consists of statements or assertions. It may be noted that not all meaningful sentences are statements for sentences can be interrogative or imperative or exclamatory. A statement is a declarative sentence, and such sentences are refered to in the terminology of logic as propositions which may not be true. It has become a truism that a statement is either true or false leading us to the consideration of a two valued logic which embodies the absolute truth. But in an axiomatic system of logic, truth does not remain absolute but it can be gradable on replacing the terms true and false by the continuous scale of probability leading to the consideration of a

many valued logic. Skolem has also developed a logic with infinitely many values for constructing an arithmetic within a system of set theory.

3. **Sentential Calculus.** In constructing sentential [or propositional] logic which is also denoted as the sentential or propositional Calculus, we use a number of sentential connectives, namely $-$, $\vee$, $\overline{\vee}$, $\cdot$, $|$, $\rightarrow$, $+\!\rightarrow$, $\leftarrow$, $\leftarrow\!+$, $\longleftrightarrow$ and $\leftarrow\!+\!\rightarrow$; and an infinite sequence of atoms [atomic formulas or sentential variables] p, q, r......s...with or withous subscripts. Starting from the atoms, we construct expressions of formulas U, V, W......in accordance with the following rules :

Formula 1. Every atom p_m is a formula [expression] ;

Formula 2. If U is a formula, then $\overline{U}$ is a formula ;

Formula 3. I U and V are formulas, then $U \vee V$, $U \cdot V$......are formulas ;

Formula 4. Nothing is a formula of sentential Calculus, except on account of rules F 1—F 3.

In order to exhibit the order in which the different sentential connectives have been applied, we must introduce a system c punctuations by means of dots and brackets.

By virtue of the above rules for constructing formulas, we can draw for every formula a pedigree, the existence of which provides a justification for the following definition of the notion of substitution.

[ʃ1] The result $\int_{W}^{p_j} p_k \Big|$ of substituting an expression W for the atom p_j in the atom p_k is

[i] W if $j=k$;

[ii] p_k if $j \neq k$.

[ʃ2] The result $\int_{W}^{p_j} \overline{U}$ of substituting the experssins W for the atom p_j in the expression U is $\overline{U'}$, where $\overline{U'}$ being the result

$\int_{W}^{p_j}$ U of substituting the expression W for the atom p_j in the expression U·

[∫3] The result $\int_{W}^{p_j}$ U v V of substituting the expression W for the atom p_j in the expression $U^1 v V^1$ is $U^1 V V^1$ U^1. V^1 being respectively the substitution results $\int_{w}^{p_j}$ U and $\int_{W}^{p_j}$ V ; similarly for other sentential connectives.

[∫4] Nothing is a substitution result except on account of the rules [∫1 — 3]. Clearly for every expression W, for every atom p_j and for every expression U, there is a unique substitution result $\int_{W}^{p_j}$ U which is always an expression [formula]

4. **Extension of sentential Calculus.** The statement calculus is limited to the structure of sentences in terms of component sentences. It cannot analyse the structure of a sentence along the subject-predicate lines that Grammarians describe.

In other words, the statement calculus does not break down a sentence into sufficiently fine constituents for most purposes. With the addition of three additional logical notions : terms, predicates and quantifiers, sentential calculus is extended to a predicate calculus in which much of everyday and mathematical language can be symbolized. It is also possible to axiomatize the predicate calculus.

5. **Axiomatic theories.**—To day, axiomatic theories usually presented is essentially the same that Euclid began his development of geometry—by listing the primitive notions and the axioms of the theory.

In its technical sense the word "theory" is applied to two sets of statements, of which one is a distinguished subset of the other. The entire set of statements defines the subject matter of the theory. In sciences, apart from mathematics, the members of the distinguished subset are those statements which are classified as true statements about the real world, with experiment the ultimate basis for the classification. In axiomatic theory the notion of truth plays no role whatsoever in the determination of the distinguished subset. Its members which are called provable statements, are defined to be those statements of the theory that can be deduced by logic alone from certain initially chosen statements called postulates or axioms. A proof is a finite column $< S_1, S_2 \ldots, S_n >$ of statements of the theory such that each S_i is either an axiom or comes from one or more preceding statements S'^s by the rule of the inferenec of the system of logic employed.

A theorem is a statement which is the last line of some proof and in particular an axiom is a theorem with a one line proof.

If Γ is a set of statements and A is a statement, then $D(\Gamma . A)$ is defined to be the set of those finite columns X of statements whose last line is A, such that each line of X is either an axiom or an element of Γ or else may be inferred from preceding statements of X by the rule of the inference of the logic employed. If, for given Γ and A, $D(\Gamma, A)$ is nonempty, then A is said to be deducible from assumption Γ, symbolized

$\Gamma \vdash A$,

and a member of (Γ, A) is called a demonstration of A from Γ. In many deductive theories we wish the axioms to be categorical i. e., the system should be adequate to decide the truth or falsity of any statement which can be formulated in the system.

An axiom system is usually accompanied by proofs of independence, consistency and completeness of its axioms and presumably should also be accompained by proofs of the independence, consistency and completeness of its primitive terms. Frequently a set of axioms, and terms admits alternative equivalent axiom systems : i. e., nonidentical partitionings of the statements into axioms and theorems, respectively; and of the terms into primitive and defined terms. Thus a particular set of axioms and primitive terms may be thought of as a basis for a class of equivalent axiom systems.

We have considered various axiomatic systems in the following chapters. But in order to make the ideas clear, we shall develop a formulation of the Boolean propositional logic. An ideal I of B is a subset of B such that for all α and β in I the element $\alpha+\beta \div I$, and for all $\alpha \div I$ and $\gamma \div B$, the element $\alpha.\gamma \div I$.

The dual concept is the notion of a sum ideal i. e. $\int$ is a sum ideal iff $\int \sqsubset B$ and $\alpha, \beta \div \int$ implies $\alpha \cdot \beta \div \int$, and $\alpha \div \int$, $\gamma \div B$ implies $\alpha+\gamma \div \int$. Equivalently $\int$ is a sum ideal of B iff $\int \sqsubset B$ and $\alpha \div \int$ and $a \sqsubset \beta$ imply $\beta \div \int$. By a maximal ideal (sum ideal) we mean a proper ideal (sum ideal) which is contained in no other ideal (sum ideal).

Consider an ordered quadruplet, $(\Sigma, \Sigma_t, \rightarrow, -)$, where Σ is a class of elements, Σ_t is a sub class of Σ, $\rightarrow$ is binary operation and — is a u nary operation in Σ such that :

1. If p is in Σ_t, then $-p$ is on Σ;
2. If p and q are in Σ, then $p \rightarrow q$ is a uniquely determined element of Σ_t
3. If p is in Σ, then $-p$ is a unique element of Σ
4. If p, q and r are in Σ, then $[p \rightarrow (q \rightarrow r)] \rightarrow (p \rightarrow q) \rightarrow (p \rightarrow r)]$ is in Σ_t
5. If p and q are in Σ, then $p \rightarrow (q \rightarrow p)$ is in Σ_t
6. If p and q are in Σ, then $(-p \rightarrow -q) \rightarrow (q \rightarrow p)$ is in Σ_t
7. If p and $p \rightarrow q$ are in Σ_t, then q is in Σ_t.

The system may be interpreted as follows :

Σ is the class of propositions,

Σ_t is the class of true propositions, $p \rightarrow q$ is the proposition that if p, then q; and $-p$ is the proposition that it is false that p.

The following deduction theorems are stated without their proofs.

THEOREM 1. If $B = (\Sigma, \rightarrow, ')$ is a Booleanalgebra and we define $p \rightarrow q$ as $(p \cap q')'$ and $-p$ as p' then for the subclass Σ_t of Σ to satisfy axioms 4 – 7 it is necessary and sufficient that Σ_t be a sum ideal.

2. A necessary and sufficient condition that a propositional logic be categorical is that Σt be a maximal ideal.

3. If Γ is a finite subclass, $\{p_1, p_2, \cdots p_m\}$ of Σ, then q is a consequence of Γ iff $\vdash p_1 \cap p_2 \cdots\cdots \cap p_m \rightarrow q$.

4. If Γ is a nonempty subclass of Σ, then q is a cousequence of Γ iff there are elements $p_1, p_1, \cdots p_m$ of Γ, such that $\vdash p_1 \cap p_2 \cdots \cap p_m \rightarrow q$

5. The element q is a cousequence of the null class iff $\vdash q$

6. The element q is a consequence of p iff $\vdash p \rightarrow q$

7. The class $\overset{*}{\Gamma}$ of consequences of Γ is the smallest sum ideal containing both Γ and Σ_t.

A subclass Γ of Σ is called a deductive system if it contains all its consequences i.e. $\overset{*}{\Gamma} \sqsubset \Gamma$. But by 7 $\overset{*}{\Gamma} \sqsubset \Gamma$, so that $\Gamma = \overset{*}{\Gamma}$ and this implies that Γ is a sum ideal containing Σt. Conversely if Γ is a sum ideal containing Σt, then $\Gamma = \overset{*}{\Gamma}$

8. The Concept of a deductive system coincides with that of a sum ideal containing Σt A deductive system is the set of all consequences of some given class of propositions.

The truth values form a two element Boolean algebra with respect to the operations $\cap$ and $'$ and may therefore, be identified with the elements O and I. Then the truth value is a homomorphism of the given Boolean algebra B on to the two element Boolean algebra defined by $v(p') = v(p)'$. $v(p \cap q) = v(p) \cap v(q)$.

9. Let Σt be the class of propositions with the truth value truth'' i-e $v(p) = 1$ for all $p \in \Sigma t$. Then Σt is a maximum sum 'ideal (M S I)

10. Conversely, if Σt is a MSI and if we define v by

$$v(p) = \begin{cases} I \text{ if } p \in \Sigma t \\ 0 \text{ if } p \notin \Sigma t \end{cases}$$

then v is a homomorphism of the algebra of propositions to the two element Boolean alegebra.

11. An assignment of truth values is possible in a Boolean propoiitional logic iff the logic is Categorical.

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1. THE PROPOSITIONAL CALCULUS

1. Propositional Operations.

To begin with, we shall lay down careful rules of usage for certain key words as referred to in the introduction : 'not', 'and', 'not both— and—', 'or', 'neither—nor—', 'if—then—' '—if—', '—but not—', 'not—but—,' 'if and only if', '—or—but not both' and '—or—according as—or not—'. These key words are used to construct molecular sentences out of atomic sentences ; and are, therefore, called propositional operations or sentential connectives. 'Not' is a monary (unary) operation, whereas others are binary and ternary operations. The variables a, b, c,......with suffixes (if needed) will be used for propositions.

1·1. Negation, Conjunction and non-conjuction. The word 'not' is used to negate a proposition ; and is denoted by '—'. If a is a proposition "Krishna loves Radha", then $\bar{a}$ is the proposition "Krishna does not love Radha".

The word 'and' is used to conjoin two proposition to make a single proposition, which is called the conjunct of a and b ; and is denoted by $a.b$, or $a \& b$, or simply ab. Since ab has properties analogous to the product of two numbers in arithmetic, it is called the logical product of 'a and b' which are called the factors of the product.

The words 'not both—and—' are used for the denial of of the conjunct of two propositions, which is called the non-conjunct of two propositions say, 'a and b and is denoted by $\overline{a.b}$ or Sheffer's stroke $a|b$.

1·2. Disjunction and non-disjunction. The word 'or' is used to obtain the disjunction of two sentences. It is used in two distinct senses : inclusive (either—or—or both) and is also denoted as 'and/or' in juristic language and exclusive (either—or—but not both). The latin words 'vel' and 'aut' respectively denote the 'inclusive or' and 'exclusive or', for which the signs $\vee$ and $\wedge$ are used. Since $a \vee b$ has properties analogous to the sum of two numbers, it is also called the logical sum of a and b, which are called the summands ; and, therefore, $a \vee b$ is also written as $a + b$.

The words 'neither—nor—' are used for the denial of the inclusive disjunct of two propositions and is denoted by '$\bar{\vee}$'

1·3. Implication, Converse-Implication, non-implication and converse non implication. The words 'if—then—' are used to obtain the conditional proposition from two propositions. The conditional proposition formed from a and b is written as $a \rightarrow b$, where a is called implicans (hypothesis, antecedent) and b implicate (conclusion, consequent). $a \rightarrow b$ is an analogue of $a \leqslant b$ in elementary algebra. Many variants of $a \rightarrow b$ are in common use occuring in mathematical reasoning, the most common of which are :

'b is a necessary condition for a'.
'a is a sufficient condition for b.'
'b if a·'
'b provided that a.'
'a only if b.'
'a implies b,'

This type of implication is sometime called as material implication.

The words '—if—' are used for the interchange of the hypothesis and conclusion in an implication $a \rightarrow b$, which is denoted by $a \leftarrow b$.

The words'—but not—' are used for the denial of the conditional statement formed from two propositions, which is called non-conditional or non-implicational, and is denoted by. $\not\rightarrow$.

The words 'not—but—' are used for the denial of the converse conditional proposition from two propositions *a* and *b*, and is denoted by '$\nleftarrow$'.

1·4. Equivalence and non-equivalence. We use the words 'if and only if' or the variants in mathematics "—necessary and sufficient condition for—," to obtain a biconditional proposition. A biconditional proposition formed from *a* and *b* is also called an equivalence and is denoted by $a \longleftrightarrow b$, where *a* and *b* are called the left and the right members (terms) of the equivalence.

The words '—is not equivalent to—' or their variants '—or —but not both' are used for the denial of a biconditional sentence which is called non-biconditional or non-equivalence and is denoted by $\nleftrightarrow$.

1·5. Conditioned disjunction. The words '—or—according as—or not—' are used to obtain a conditioned disjunction from three propositions. The conditioned disjunction formed from *a*, *b* and *c* is denoted by $[a, b, c]$ and is read as "*a* or *c* according as *b* or not *b*", which is *a* ternary operation.

Exercise I

A. Use the word 'but' a conjuction.

B. Formulate in symbols, using letters to stand for atomic sentences :

1. Krishna loves Radha aud Radha loves Krishna.

2. You cannot both have your meal and eat it. a : you have your meal. *b* : you eat it. $a \cdot \overline{b}$ or $a|b$.

3. There will be rains in summer, or there will not be. $a \vee \overline{a}$. where a: there will be rains in summer.

4. Either the examiner did not send the answer packet, or it has been lost in the mail.

5. She cannot both sing and not dance.

$\overline{a \cdot \overline{b}}$, where *a* : she sings. *b* : she dances.

6. Neither the registrar nor the vice-chancellor was in the Senate hall.

$a\overline{\vee}b$, where a : the registrar was in the Senate hall b: the Vice chancellor was in the senat hall.

7. If Krishna or Ahad comes, we shall play chess $a \vee b \rightarrow c$.

8. In the same circle if the two chords are unequal, they are unequally distant from the centre.

9. When triangles are congruent, they have equal angles and equal areas; and when traingles have equal angles and equal areas they are congruent. $a \longleftrightarrow bc$.

2. **Parentheses and dots.** Parentheses and brackets of various kinds are used in symbolic logic to indicate the relative dominance of various sentential connectives. For example in

† $p(q \longleftrightarrow ((r \rightarrow s) \vee t))$, the conjunction is the dominant operation of the formula.

The parentheses can be replaced by dots, where the dot notation does not stand for a logical product, but strengthens whatever operations it stands by. For example the above formula can be written as $p \,.\, q \longleftrightarrow \,.\, r \rightarrow s \,.\, \vee t$ ...... (1)

A dot, on one side of an operation strengthens that side only. Any operation strengthened by a dot is stronger on that side than any operation not strengthened by a dot. If two operations are each strengthened by a dot, then they have the same strength relative to each other, as though no dot exists.
Consider the formula (1) and :

$p \,.\, q \longleftrightarrow \,.\, r \vee s \rightarrow t$.........(2)

$p \,.\, q \longleftrightarrow r \,.\, \vee s$ (3)

In the first, the dot on the left of '$\vee$' strengthens it, but the '$\longleftrightarrow$' with the dot on the right is even stronger, to the right. On the left the $\longleftrightarrow$ is not strengthened and the dot between p and q is the stronger and hence conjuction is the dominant operation in the first. In the second, the dot on the right of '$\longleftrightarrow$' strengthens it on the right so that it is stronger than '$\rightarrow$'. However '$\longleftrightarrow$' is not strengthened on the left and so the dot between p and q is stronger and here also conjunction is the dominant operation. In the third, the dot on the left of '$\vee$' strengthens it on the left and is dominant operation.

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† Generally the sign of conjunction will be omitted.

In order to strengthen an operation very much, a double dot ':' or triple dot '$\therefore$' or even a quadruple dot '::' next to the operation is placed. An operation strengthened by a double dot is, on that side, stronger than any operation with a single dot. If two operations are both strengthened by a double dot (or a triple or a quadruple), then they have the same strength relative to each other, as though no dots are peresent. Consider the formulas :

$$p \therefore q \longleftrightarrow . r \rightarrow s : \vee t \qquad (4)$$

$$p : q \longleftrightarrow : r \rightarrow s : \vee t \qquad (5)$$

In the fourth conjunction is the dominant operation and the fifth is just the repetition with double dots of the first with single dots

Conversely in a formula, each dot in succession can be replaced by a pair of parentheses, starting with the strongest dot in the formula. The strongest dot is that next to the dominant operation. Thus in the first formula, dots can successively be. replaced by pairs of parentheses as shown in the table below.

$p \,.\, q \longleftrightarrow . r \rightarrow s \,.\, \vee t$

$p \,.\, q \longleftrightarrow (r \rightarrow s \,.\, \vee t)$

$p \,.\, q \longleftrightarrow ((r \rightarrow s) \vee t)$

$p(q \longleftrightarrow ((r \rightarrow s) \vee t))$.

3. **Definitions.** The introduction of new terms as a function of known terms is called a definition. In symbolic language we

shall use the sign $\cdot\!=\!\cdot\, Df$ meaning equal by definition, or "stands for". Thus we may write :

Equilateral triangle $\cdot\!=\!\cdot\, Df$ triangle with its side equal. The terms on the left hand side of the definition is called the definiendum, the terms on the right side the definiens.

There are four traditional rules of definitions :

1. A definition must give the essence of that which is to be defined.

2. A definition must not be circular.

3. A definition must not be in the negative, when it can be in the positive.

4. A definition must not be expressed in figurative or obscure language.

The use of parentheses, dots and definitions is not a part of symbolic logic but means of translating a variety of linguistic devices.

Exercise 2.

A. Replace parentheses by dots in the following :

1. $p \rightarrow (q \rightarrow p)$.
2. $[\mathrm{p} \rightarrow (q \rightarrow r)] \rightarrow [q \rightarrow (p \rightarrow r)]$.
2′ $[(p \rightarrow q) \rightarrow p] \rightarrow p$
2″ $[p \rightarrow (p \rightarrow q)] \rightarrow (p \rightarrow q)$.
3. $(p \rightarrow q) \rightarrow [(q \rightarrow r) \rightarrow (p \rightarrow r)]$.
4. $(p \longleftrightarrow q) \rightarrow (p \rightarrow q)$.
5. $(p \longleftrightarrow q) \rightarrow (q \rightarrow p)$.
6. $(p \rightarrow q) \rightarrow [(q \rightarrow p) \rightarrow (p \longleftrightarrow q)]$.
7. $[q \rightarrow \bar{p}\ ] \rightarrow [p \rightarrow q]$.
8. $[p \rightarrow (q \rightarrow r)] \rightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)]$.

B. Replace the dots with parantheses in the following :

1. $p \to q . \leftarrow \to . \overline{p} \vee q.$
2. $pq : \longleftrightarrow : p \to . p \to q.$
3. $p \longleftrightarrow q . \longleftrightarrow . p \to q . \overline{p} \to \overline{q}.$
4. $p \to . q \longleftrightarrow r : \to : p \to q . \to . p \to r.$

C. 1. What are the traditional rules of a 'definition'.

2. Define the relations '$\angle$' and left factor (l. f.) in the domain of ordinal numbers.

($a \angle b \ .=\cdot Df$ there exists a unique ordinal c such that $a+c=b$.

$a \, l. f. \, b \cdot =\cdot \ Df$ there exists a unique non zero ordinal d such that $ad=b$.

3. **Truth tables and matrices**. To propositions are assigned "truth values" ; the truth value of a proposition being truth 't', if it is true, and falsehood 'f' if it is false. The study of propositional formulas with special reference to their truth and falsity, constitute the Propositional Calculus. The truth functional rules of usage for negation, conjunction and other operations may be summarised in a tabular form.

The basic truth tables tell us at a glance, under what circumstances the negation of a proposition is true. if we know the truth or falsity of a proposition, similarly for the conjunction and another operations of two propositions.

a	$\overline{a}$	a	b	ab	a/b	$a \vee b$	$a \overline{\vee} b$	$a \to b$	$a \nrightarrow b$	$a \leftarrow b$	$a \nleftarrow b$	$a \longleftrightarrow b$	$a \nleftrightarrow b$ or $a \vee b$
t	f	t	t	t	f	t	f	t	f	t	f	t	t
f	t	t	f	f	t	t	f	f	t	t	f	f	t
		f	t	f	t	t	f	t	f	f	t	f	t
		f	f	f	t	f	t	t	f	t	f	t	f

A truth table for an operation can be transformed into a new truth table for another operation by the following device. Truth tables of $\vee$, $\rightarrow$, $\leftarrow$, $\leftrightarrow$ and $|$ can respectively be transformed into the tables of $.$, $\leftarrow +$, $+\rightarrow$, $\wedge$ and $\overline{\vee}$ and vice versa by interchanging simultaneously *t* into *f* in each table. The truth tables of $|$, $\overline{\vee}$, $\leftarrow +$, $+\rightarrow$ and $\wedge$ can be obtained from the truth tables of the corresponding $.$, $\vee$, $\leftarrow$, $\rightarrow$, $\leftrightarrow$ and vice versa by interchanging *t* into *f* simultaneously in the corresponding column of each table only.

The truth table of the ternary operation $[a, b, c]$ is finally constructed.

a	b	c	$[a, b, c]$
t	*t*	*t*	*t*
t	*t*	f	*t*
t	f	*t*	*t*
t	f	f	f
f	t	*t*	f
f	*t*	f	f
f	f	*t*	t
f	f	f	f

The truth table of $[a, b, c]$ can be transformed into that of' $[c, b, a]$ and conversely by interchanging simultaneously '*t* in to 'f'.

Let us denote the truth function of a proposition by the symbol 0, if it is false and 1 if it is true. Then we can construct matrix for an operation, whose elements will be either 0' or '1' as follows.

$$\begin{array}{c|cc} \rightarrow & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 0 & 1 \end{array}$$

The matrix is used in the following manner. To find what element $0 \rightarrow 1$ is, say, we look at the entry occurring in the second row and third column and find $0 \rightarrow 1 = 1$. We note on the other hand that $1 \rightarrow 0$ (occurring in the third row and second column)$=0$. Similarly the matrix of other operations can be constructed. Analogous remarks hold good in the transformation of a matrix of one operation into a new matrix for the other operation.

There are propositions which are not two valued, i. e, as being true, or false. It is, therefore necessary to generalise truth values of a proposition by replacing the terms 'true' and 'false' by a continuous scale of probability.

By giving an arithmetic model of the truth values, Lukasiewicz[1] has developed many valued propositional logic.

He introduced n different truth values : $0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots \frac{n-2}{n-1}, 1$

instead of the only truth values O and 1, for every atomic sentence. Thus in $(n+1)$ valued logic, the truth values of an atomic sentence will be $O, \frac{1}{n}, \frac{2}{n}, \ldots \ldots\ldots \frac{n-1}{n}, 1$

The general n valued matrices of the operations are characterised by the following formulas.

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[1] LukasiewiCz and A. Tarski, Untersuch ungen über den Assagen kalkül, Comptes rendus des Seances de la SoCo. des Sciences et des Letters de varsovie, 23 (1933) 1 – 21 23(1930) 1 – 21

$$\mathrm{Tr}\,(\bar{x}) = 1-x$$

$$\mathrm{Tr}\,(xy) = \begin{cases} x, \text{ if } x \leqslant y \\ y, \text{ if } x > y \\ , \text{ i. e, } \min\,(x, y) \end{cases}$$

$$\mathrm{Tr}\,(x \vee y) = \begin{cases} y, \text{ if } x \leqslant y \\ x, \text{ if } x > y \\ , \text{ i. e., } \max\,(x, y) \end{cases}$$

$$\mathrm{Tr}\,(x \rightarrow y) = \begin{cases} 1, \text{ if } x \leqslant y \\ 1-x+y, \text{ if } x > y \\ , \text{ i. e, } \min\,(1,\, 1-x+y) \end{cases}$$

$$\mathrm{Tr}\,(x \leftarrow\rightarrow y) = \begin{cases} 1-y+x, \text{ if } x \leqslant y \\ 1-x+y, \text{ if } x > y \\ , \text{ i. e } \min\,(1-y+x,\, 1-x+y). \end{cases}$$ where the symbols

x and y on the right stand for Tr (x) and Tr (y) for brevity, if there be no ambiguity; and the value of each truth function is $\frac{k}{n-1}$, $0 \angle k \angle n-1$.

Matrices for other operations can analogously be defined.

Rosser [2] has also developed a n valued Propositional Calculus, in which the truth value function of a proposition is allowed to take any truth value : 1, 2, 3, ...n the first m of these 1, 2......m (where $2 \leqslant m < n$) being called designated truth values and the remainders as non-designated.

The matrices for the connectives are characterised as follows.

$$\mathrm{Tr}(\bar{\mathrm{x}}) = n-x+1$$

$$\mathrm{Tr}(x \rightarrow y) = \begin{cases} 1 \text{ for } x \geq y \\ 1+y-x \text{ for } x \angle y \\ , \text{ i. e. } \max\,(1,\, 1+y-x) \end{cases}$$

$$\mathrm{Tr}\,(x \cdot y) = \max(x, y)$$

$$\mathrm{Tr}\,(x \vee y) = \min(x, y),$$

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[2] J.B. Rosser, Axiom Schemes for m valued Propositional Calculi, J. Symb. Logic, 10 (1945) 61—82.

where x and y have the analogous meaning. Matrices for other operations can be defined analogously.

G. Carl Hampel [3] has topologically interpreted the arithmetic model of Lukasiewicz so that instead of the truth values, only relations of order including equality are taken into account.

Exercise 3

1. Which of the truth functional connectives is an approximate synonym of the connective 'unless ?

2. If $p \rightarrow q$ is true, what can be said about the truth value of $p \leftarrow q$ and of $p \leftarrow + \rightarrow q$?

3. If $p \leftarrow \rightarrow q$ is true, what can be said about the truth value of $p \wedge q$? [Hint $\wedge$ is non-equivalence]

4. Given a matrix of '→' and '—' in three valued propositional logic, where o (false), $\frac{1}{2}$ (possible) and 1 (true),

$\rightarrow$	0	$\frac{1}{2}$	1	—
0	1	1	1	1
$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$
1	0	0	1	0

Can you construct matrices for the other operations ?

4. Tautologies. A formula is called a tautology (universally valid) if it is assigned one of the designated truth values, independent of the truth values of the atomic propositions of which it is composed. The class of all tautologies in n valued logic will be denoted by T_n. A tautology of T_2 is always true having its truth value as 1. It can easily be shown that every tautology of T_n is a tautology of T_2 but not conversely. In this section we shall only consider T_2.

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[3] G. Hampel, A purely topological form of non-Aristotelian logic J. Symb. logic 2 (1937) 97--112

A tautology can be verified by means of a truth table. The table of $a \to b. \longleftrightarrow. \bar{a} \vee b.$ is considered.

a	$\to$	b	$\longleftrightarrow$	$\bar{a}$	$\vee$	b
t	t	t	t	f	t	t
t	f	f	t	f	f	f
f	t	t	t	t	t	t
f	t	f	t	t	t	f

In this table the second column is obtained from the first and third by using the implication table; the fifth column is obtained from the first by using the negation table, the sixth column is obtained from the fifth and seventh by using disjunction table and the fourth column is obtained finally from the second and the sixth by using equivalence table. Since each row of the fourth column has the entry '*t*', we conclude that the formula is a tautology. It shoud be noted that a formula is a tautology if and only if every entry in the final column is '*t*'. The letter 'f' in a single row of the final column is sufficient to guarantee that the formula being verified is not a tautology.

In general if there are n distinct variables in a tautology, then there are 2^n possible combinations of the truth values and thus there are 2^n rows to the truth table. In the above tautology there are 2^2 rows to the truth table. As the notion of tautological implication and equivalence is important in the theory of inference (as considered in the next section), we shall introduce such tautologies.

A proposition p is said to tautologically imply a proposition q if and only if the conditional '$p \to q$' is a tautology. For example pq tautologically implies q, for it can easily be verified that $pq \to q$ is a tautology.

When two propositions tautologically imply each other, they are said to be tautologically equivalent; for example $p \to q. \longleftrightarrow. \bar{q} \to \bar{p}$

Tautologies of propositional calculus are given in the following three tables.

A. Tautologies of excluded middle and contradiction

1. Law of Excluded middle $a \vee \bar{a}$
2. Law of contradiction $a \, . \, \bar{a}$

B. Tautological implications

1. Law of Simplification or implication from both to any $ab \rightarrow a$
2. Law of arbitrary addition of an implication $\begin{cases} a \rightarrow . \; b \rightarrow a \\ \bar{a} \rightarrow . \; a \rightarrow b \end{cases}$
3. Law of addition of a summand in the implicate $a \rightarrow b . \rightarrow . \; a \rightarrow b \vee c ; \; p \rightarrow . p \vee q$

4 Law of addition of a factor in the implicans $a \rightarrow b . \rightarrow . \; ac \rightarrow b$

5 Law of dropping a summand in the implicans $a \vee c \rightarrow b . \rightarrow . \; a \rightarrow b$

6. Law of dropping a factor in the implicate $a \rightarrow bc . \rightarrow . \; a \rightarrow b$
7. Law of self distributive $c \rightarrow . \; a \rightarrow b \; : \rightarrow : \; c \rightarrow a . \; \rightarrow . \; c \rightarrow b$
8. Law of transitivity of implication $a \rightarrow b . \; b \rightarrow c : \rightarrow . \; a \rightarrow c$
9. Law of transitivity of equivalence $a \longleftrightarrow b . \; b \longleftrightarrow c : \rightarrow . \; a \longleftrightarrow c$
10. Law of Exportation $ab \rightarrow c . \rightarrow : \; a \rightarrow . \; b \rightarrow c$
11. Law of importation $a \rightarrow . \; b \rightarrow c : \rightarrow . \; ab \rightarrow c$
12. Law of absurdity $a \rightarrow \bar{b} b \; . \rightarrow \bar{a}$
13. Law of detachment $a . \; a \rightarrow b : \rightarrow b$
    (Modus Ponens)
14. Modus tollendo tollens $\bar{b} . \; a \rightarrow b : \rightarrow \bar{a}$
15. Modus tollendo ponens $\bar{a} . \; a \vee b : \rightarrow b$
16. Modus ponendo tollens $a . \; a \wedge b \rightarrow \bar{b}$

C. Tautological Equivalence

1. Law of double negation $a \leftrightarrow \overline{\overline{a}}$

2. Law of dissolution or Law for biconditional $\left.\begin{matrix} a \leftrightarrow b : \leftrightarrow : a \to b . b \to a \\ a \leftrightarrow b : \leftrightarrow : ab \lor \bar{a}\,\bar{b} \end{matrix}\right.$

3. Law of negation for Equivalent $\overline{a \leftrightarrow b} . \leftrightarrow . a \leftrightarrow \bar{b}$

4. Law of negation of equivalent members

$$a \leftrightarrow b . \leftrightarrow . \bar{a} \leftrightarrow \bar{b}$$

5. Law of contraposition $a \to b . \leftrightarrow . \bar{b} \to \bar{a}$

6. De Morgan's laws $\left\{\begin{matrix} \overline{ab} . \leftrightarrow \bar{a} \lor \bar{b} \\ \overline{a \lor b} . \leftrightarrow . \bar{a}\, b \end{matrix}\right.$

7. Commutative laws $\left\{\begin{matrix} ab \leftrightarrow ba \\ a \lor b . \leftrightarrow . b \lor a \end{matrix}\right.$

8. Associative laws $a . bc . \leftrightarrow . ab . c$

$$a \lor . b \lor c . \leftrightarrow . a \lor b . \lor c$$

9. Distributive laws $\left.\begin{matrix} a . b \lor c . \leftrightarrow . ab \lor ac \\ a \lor bc . \leftrightarrow . a \lor b . a \lor c \end{matrix}\right.$

10. Law of dropping of a true factor $a . b \lor \bar{b} . \leftrightarrow a$

11. Law of dropping of a false summand $a \lor . b \bar{b} : \leftrightarrow : a$

12. Law of merging of implications $\left\{\begin{matrix} a \to b . a \to c : \leftrightarrow . a \to bc \\ a \to c . b \to c : \leftrightarrow . a \lor b \to c \\ a \to b . \lor . a \to c : \leftrightarrow . a \to b \lor c \\ a \to c . \lor . b \to c : \leftrightarrow . ab \to c \end{matrix}\right.$

Exercise 4.

A. Verify the following tautologies :

1. $a_1 \to b . a_2 \to b . a_3 \to b , \ldots\ldots a_n \to b . \leftrightarrow . a_1 \lor a_2 \ldots a_n \to b$

2. $a \rightarrow b. \vee a_2 \rightarrow b. \vee . a_3 \rightarrow b.$

$\vee \ldots\ldots \vee . a_n \rightarrow b . \longleftrightarrow . a_1 \, a_z \ \ldots\ldots \ a_n \rightarrow b$

3. $\overline{a_1 a_2 a_3 \ldots\ldots a_n} \ . \leftarrow \rightarrow . \ \overline{a_1} \ a_2 \vee \overline{a}_3 \ldots\ldots \vee \overline{a^n}$

4. $\overline{a_1 \vee a_2 \vee a_3 \ldots\ldots \vee a_n} \longleftrightarrow \overline{a_1} \ \overline{a_2} \ \overline{a_3} \ldots\ldots \overline{a_n}$

B. Let a, b, and c be any three distinct atomic sentences, decide by truth table which of the following sentences are tautologis.

1. $a \vee \overline{a}$
2. $ab \rightarrow a \vee c$
3. $ab \rightarrow : a . \longleftrightarrow . \ b \vee c$
4. $a \rightarrow b . \rightarrow . \ b \rightarrow a$

C. If a and b are distinct atomic propositions, the proposition ab, tautologically implies which of the following ?

1 a

2. b

3. $\overline{a} \vee b$

4 $b \rightarrow \overline{a}$

5$a \longleftrightarrow b$

D. If a and b are distinct atomic propositions, the proposition $a \rightarrow b$ tautologically implies which of the following ?

1 a 2. $\overline{b} \rightarrow \overline{a}$

3. $b \rightarrow a$ 4. $\overline{a} \, b$

E. If a and b are distinct atomic propositions, the proposition $a \longleftrightarrow b$ is tautologically equivalent to which of the following ?

1. $ab \vee \overline{a} \ \overline{b}$ 2. $\overline{a} \longleftrightarrow \overline{b}$

3. $a \longleftrightarrow \overline{b}$ 4. $\overline{b} \longleftrightarrow \overline{a}$

5. Theory of inference. Various works on traditional logic treat of certain kinds of inferences, which are some of the useful tautological implications.

5·1. Modus ponens. A tautological implication frequently used in deriving conclusions is the law of detachment. (in latin modus ponens) $a. \, a\to b :\to . \, b$, which corresponds to the rule that from a and $a\to b$, we may infer b; and is also written as $a\to b$

$$\frac{a}{\quad b} \quad (1)$$

The first line $a\to b$ is called the major premise (formula), the second line the minor premise while the third the conclusion. An example is given by the propositions :

If hydrogen burns in air, water is produced.
Hydrogen is burnt in air
water is produced.

The above rule is called fundamental, since other rules are devided from it which are called secondary

5·2 Modus tollens : $\overline{b}. \, a\to b\to \overline{a}$ is called the rule of negative inference or modus tollens. An example is given by the following propositions.

If hydrogen is burnt in air, water is produced.
Water is not produced
Hydrogen is not burnt in air.

It is also derived from modus ponens :

$$\overline{b} \; a\to b . \longleftrightarrow . \, \overline{b} \cdot \overline{b} \to \overline{a} \to \overline{a}$$

These two rules are known as Hypothetical syllogism

5·3 Modus tollendo ponens $\overline{a} \, . \, a \vee b . \to . \, b$ is called inclusive disjunctive inference (Modus tollendo ponens), which can also be derived from modus ponens; $\overline{a}. \; a \vee b . \longleftrightarrow \overline{a} \cdot a \to b \to b$

5·4 Modus ponendo tollens. $a \, . \, a \wedge b \to \overline{b}$ is called exclusive disjunctive inference (Modus ponendo tollens); which can be derived from modus ponens in two steps.

$a \wedge b \longleftrightarrow a \vee b.\ \overline{ab.} \longleftrightarrow.\ \ a \rightarrow b.\ a \rightarrow \overline{b} \rightarrow . a \rightarrow \overline{\overline{b}}$

$a \wedge b.\ a \wedge b \rightarrow a \rightarrow\ b : \rightarrow . a \rightarrow \overline{b}$

Again, $a . a \rightarrow \overline{b}\ : \rightarrow . \overline{b}$

The above rules are called disjunctive syllogism

5·5 Chain rule of inference. $a_1 \rightarrow a_2 .\ a_2 \rightarrow a_3 .\ a_3 \rightarrow a_4$ ………… $a_{n\text{-}1} \rightarrow a_n\ : \rightarrow . a_1 \rightarrow a_n$ is called the chain rule of inference if each of the implications is true. If in addition a_1 is true, a_n can be asserted. The chain rule of inference is used in deriving dilemma. It has the form

$a \vee b.\ a \rightarrow c.\ b \rightarrow d : \rightarrow . c \vee d$, which can be derived from the chain rule : --(1)

$\overline{c} \rightarrow \overline{a}.\ \overline{a} \rightarrow b.\ b \rightarrow d. \rightarrow .\ \overline{c} \rightarrow d. \longleftrightarrow .\ c \vee d$

This dilemma is called complex constructive. If in (i) the sentences c and d are identical, the dilemma is called simple constructive and has the form :

$a \vee b.\ a \rightarrow c.\ b \rightarrow c . \rightarrow .\ c$ (ii)

If the disjunction has the form $a \vee \overline{a},$ in the premiss, it can be omitted because of its tautological character and it assumes the shorter form : $a \rightarrow c.\ \overline{a} \rightarrow d. \rightarrow . c \vee d$ (iii), When, as in (ii), the two propositions 'c, and 'd' are identical, then (iii) assumes the form : $a \rightarrow c.\ \overline{a} \rightarrow c : \rightarrow c$ (iv) and is called an analytic dilemma.

There is a second type of dilemma called complex destructive, which has the form :

$a \rightarrow c\ .\ b \rightarrow d\ .\ \overline{c} \vee d : \rightarrow .\ \overline{a} \vee \overline{b},$ which can be derived by the chain rule analogously (v).

When, in particular the propositions a and b are identical, the dilemma is called simple destructive and has the form :

$a \rightarrow c\ .\ a \rightarrow d.\ \overline{c} \vee d : \rightarrow \overline{a}$ (vi)

There is another rule of inference called the rule of ,,substitution'' which we shall use frequently in the next sections.

5·6 Rule of substitution. It is permissible to assert a formula resulting from a tautology when a propositional expression, which is arbitrarily chosen, is substituted in the place of any free elementary propositional variable, provided that the substitution is made in all places where the original variable occurs. For example, substitution of $p \vee q$ for a in the formula : $a \vee a. \rightarrow a$ gives $p \vee q. \vee .p \vee q : \rightarrow . p \vee q$, which is also a tautology.

Exercise 5

1. Give an example from an elementary geometary that if the propositions 'p' and 'q' are proved, then pq is asserted.
2. What is the rule of equivalent inference ?

(Hint $a.a \longleftrightarrow b : \rightarrow b$)

3. Given : $p \rightarrow q \vee r$

$\bar{q}\ \bar{r}$

Derive : $\bar{p}$

Hint $(p \rightarrow q \vee r. \longleftrightarrow . \bar{q}\overline{\vee} r \rightarrow \bar{p} \ . \longleftrightarrow . \ \bar{q}\ \bar{r} \rightarrow \bar{\mathrm{p}}$

$\bar{q}\ \bar{r} \cdot \bar{q}\ \bar{r} \rightarrow \bar{p} \ \rightarrow . \ \bar{p}$)

4. Given : $p \vee \cdot q \rightarrow r : q\ \bar{r} \vee \bar{s}. : s$

Derive p.

6. Axiomatisation of Propositional Calculus

Given a set of tautologies, it is possible to choose a minimum number of them, called as axioms and derive the rest in terms of these axioms. As the choice of axioms is not unique, there can be different axiom systems (logistic systems).

Consider a logistic system Σ, which has implication→and falsehood 'o' as primitive, substitution and modus ponens as the rules of inference and which has as its axioms law of affirmation of the implicate, self distributive law of implication and the law of negation, as listed below.

6·1 Axioms.

1. $a \rightarrow . \; b \rightarrow a$
2. $c \rightarrow . \; a \rightarrow b : \rightarrow : c \rightarrow a . : \rightarrow . \; c \rightarrow b$
3. $a \rightarrow o . \rightarrow o : \rightarrow a$

Theorems are derived by substitution and modus ponens. For the sake of brevity, the sign $\vdash$ will be used to express that a formula is a theorem in a logistic system. For example $\vdash . o \rightarrow p$ is read as '$o \rightarrow p$' is a theorem.

4· $\vdash . o \rightarrow a$. The proof of the theorem is the following sequence of formulas.

By substitution of $a \rightarrow o . \rightarrow o : \rightarrow a$ for a and o for b in 1 :

$$a \rightarrow o . \rightarrow o : \rightarrow a \because \rightarrow : o \rightarrow : a \rightarrow o \rightarrow o \rightarrow a .$$

By 3 and modus ponens :

$$o \rightarrow : a \rightarrow o \rightarrow o . \rightarrow a$$

By simultaneous substitution in 2 :

$$o \rightarrow . \; a \rightarrow o \rightarrow o \rightarrow a : \rightarrow : o \rightarrow . \; a \rightarrow o \rightarrow . o \rightarrow . o \rightarrow a$$

By modus ponens.

$$o \rightarrow . \; a \rightarrow o \rightarrow o . \rightarrow : o \rightarrow a$$

By simultaneous substitution in 1 :

$$o \rightarrow . \; a \rightarrow o \rightarrow o$$

Hence by modus ponens :

$$o \rightarrow a$$

This theorem is often stated : "A false proposition implies every proportion"

The following theorems can analogously be proved.

5. $\vdash a \rightarrow b . \rightarrow : b \rightarrow c . \rightarrow . \; a \rightarrow c$
6. $\vdash a \rightarrow b . \rightarrow a : \rightarrow . a$ Peirceis' Law

Let Σ_1 be the logistic system which has the same primitive and rules of inference as Σ and which has its axioms 1 and the theorems 4, 5 and 6 of Σ. Then Σ and Σ_1 are equivalent systems in the sense that the axioms of Σ can be deduced from those of Σ_1 and conversely. The verifications of equivalence is left as an exercise.

The independence of the axioms is exhibited.

6.2 Independence : An axiom A of a logistic system is called independent if in the logistic system obtained by omitting A from among the axioms $\overline{\vdash A}$. An equivalent definition of independence is also given. An axiom is called independent if there is some theorem which cannot be proved without that axiom.

Independence of each axiom is established by means of the following matrices of 3 valued logic of which 1 is only designated truth value and 3 is assigned the primitive symbol o (falsehood) as a value.

1

→	1	2	3
1	1	3	3
2	3	3	1
3	1	1	1

II

→	1	2	3
1	1	2	3
2	1	1	2
3	1	1	1

III

→	1	2	3
1	1	2	3
2	1	1	3
3	1	1	1

It can easily be verified that matrix I satisfies axioms 2 and 3, but not the first. For if we choose for $a=2$ and $b=3$, the axiom 1 obtains the value 3. Analogous remarks hold good for other matrices.

Decision problem Consistency and completeness of Σ will also be established.

The two rules of inference of Σ preserve tautologies, in the sense that if the premiss or premisses are tautologies, the conclusion is a tautology. It can easily be proved that every theorem of Σ is a tautology and conversely. The verification of the tautological character of the three axioms is trivial.

6·3 Decision problem. A decision problem of a logistic system is the problem of finding a method by which, given any formula it is possible to decide in a finite number of steps, whether or not it is a theorem; and if it is a theorem to obtain a proof of it.

The tautological character of a formula can easily be decided by means of a truth table in a finite number of steps. The effective procedure for recognising tautologies and the effective proofs of the corresponding theorem constitute a solution of the decision problem of Σ.

6·4 Consistency. A system is called consistent if every formula of that system is not provable in that system. This amounts to saying that no two formulas, one of which is the negation of the other are both provable in that system.

We shall introduce the consistency of a logistic system in the following senses. A logistic system is consistent :

(i) with respect to a given transformation if there is no sentence a such that $\vdash a$ and $\vdash a'$, where a is transformed into a';

(ii) absolutely, if not all its sentences are theorems;

(iii) in the sense of Post (with respect to a certain category of primitive symbols designated as propositional variables) if a formula consisting of a propositional variable alone is not a theorem.

THEOREM 1. Σ is consistent :

(i) with respect to the transformation of a into $a \to o$;

(ii) absolutely;

(iii) in the sentence of Post.

Proof. (1) By definition of a tautology and the truth table of $\to$, not both a and $a \to o$ can be tautologies. In fact, if a is a tautology, then $a \to o$ is a contradiction. Hence both a and $a \to o$ cannot be theorems of Σ.

(ii) Since the formula 'o' is not a tautology and therefore is not a theorem of the system and consequently Σ is absolutely consistent.

(iii) A formula consisting of a propositional variable alone is not a tautology, because its value is 'o' for the value 'o' of the variable. Therefore it is not a theorem of Σ and consistency in the sense of Post is established.

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6·5 Completeness of Σ. A logistic system is complete with respect to a given transformation (absolutely, in the sense of Post), if for every sentence a, either $\vdash a$, or the system, upon addition of a to it as an axiom, becomes inconsistent with respect to the transformation (absolutely, in the sense of Post).

THEOREM 2. Σ is complete in each of the three senses.
Proof. Let a be sentence of Σ, which is not a theorem. Then a is not a tautology, i.e, there is a system of values of the variables of a, for which the value of a is o.

If a be added to Σ as an axiom, then it is possible to infer the result of any simultaneous substitution for the variables of a. Particularly we can take one of these systems of values of the variables of a for which the value of a is o, and substitute for each variable x_i, either o or 1 according as the value of that variable is o or 1. Let b be the resulting formula.

Since b contains no variables, it can be assigned one value, and because of the manner in which b is obtained from a, it follows that this value is o. Therefore from the truth value of $\rightarrow$, $b \rightarrow o$ is a tautology. Hence $b \rightarrow o$ is a theorem of Σ, therefore, also a theorem of the system which is obtained by adding a to Σ as an axiom.

In the system obtained by adding a to Σ as an axiom, both b and $b \rightarrow$o are theorems. Therefore by modus ponens 'o' is also a theorem. Also $\vdash$o. $\vdash$o$\rightarrow p{:}\rightarrow.\ \vdash p$. Hence by substitution, we may obtain any formula whatever as a theorem including of course, with every formula 'c' also the formula $c \rightarrow$o. Thus completeness of Σ is established in each of the three senses.

6·6 Completeness of a system of connectives. All the connectives as well as the symbol, 1 (true) will be defined in terms of '$\rightarrow$' and 'o' and then the completeness of the system$\rightarrow .\ o$, will be established.

DEFINITION 1. $1 \mathrel{.{=}_{\text{Df}}.}\ o \rightarrow o$

2. $\bar{a} \mathrel{.{=}_{\text{Df}}.}\ a \rightarrow o$

3. $a \leftarrow b . = b \to a$ Df $\underline{\qquad}$

4. $a \leftarrow + - b : =$ Df $b \to a . = .$ Df $b \to a . \to o$

5. $a + \to b . =$ Df $b \leftarrow + a . = . a \to b . \to o .$

6. $a \vee b . = .$ Df $a \to b . \to b$

7. $a . b . = .$ Df $a \leftarrow + b . + \leftarrow b = b \to . b \to a . \to o : \to o$

8. $a \leftarrow\to b . =$ Df $a \to b . b \to a$

9. $a \leftarrow + \to b . =$ Df $a \leftarrow + b . \overline{\vee} . b \leftarrow + a$

10. $a \overline{\vee} b = .$ Df $\overline{a} . \overline{b .} = . a \to o . \; b \to o$

11. $a / b . = .$ Df $a \vee \overline{b .} =$ Df $a \to o . \vee . b \to o$

12. $[a, b, c] . = :$ Df $b \to a . \; \overline{b} \to c$

A system of primitive connectives for propositional calculus is called complete, if all possible truths tables of two or more columns are found among the truth tables of the resulting formula (truth tables of a formula of n variables is a finite table like the truth table of a connective). One of the connectives is called independent if, upon suppressing that particular connective, its truth table is no longer among the truth tables of the resulting formula. The symbols 't' or 'f' will be independent if upon suppressing of it, there is no longer among the formulas any one, which has the value t, or the value f respectively, for all systems of values of its variables.

The completeness of (i) conditioned disjunction, t and f has been considered by A. Church [4] by mathematical induction with respect to the number of different variables in a formula constructed from the connectives.

Suppose that among the formulas containing n variables all possible systems of values, i.e., all possible truths tables, are found. Consider a proposed system of values for a formula containing $(n+1)$ variables, i.e., a proposed $(n+2)$ column truth table T_b. Let the first column in the truth table T_b be for the variable b. Let the truth tables T'_b (T''_b) be obtained from T_b by deleting all the rows which have f (t) in the first column and then deleting the first column.

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[4] A. Church, Introduction to mathematical logic, Princeton University Press 1956.

By hypothesis of induction, formulas P and *R* exist whose truth tables are T'_b and T''_b respectively. Then the formula [P, q R] has the truth table T_b , as may be seen by reference to the truth table of conditioned disjunction. Thus the completeness of the given system follows by mathematical induction.

The independence of conditioned disjunction is trivial, for without it, there will not be formulas except those consisting of a single symbol i.e. '*f* or *t*' or a variable. The independence of *t* may be proved by reference to the last row in the truth table of conditioned disjunction, for it follows from this last row that, if a formula is constructed from conditioned disjunction and *f* (without the use of *t*), then it must have the value *f* when the value *f* is given to all its variables. The independence of *f* may analogously be established with reference to the first row in the table of conditioned disjunction.

The completeness of other systems of primitive connectives can be exihibited by defining by means of the given primitive connectives, conditioned disjunction '*t*' and '*f*'.

The completeness of (ii) implication and *f* follows from the definitions 1 and 12.

The completeness of (iii) implications and negation follows from the definition 12 toghether with any example of a tautology and of a contradiction : say $p \rightarrow p$ and $p \leftarrow + p$. which stand for '*t*' and '*f*' respectively.

The independence of '*f*' in Σ and '—' in Σ_2 is obvious because no formula constructed from '→' alone can be a contradiction. The independence of →in each case is also trivial because of the very restricted class of formulas which may be constructed without implication.

The completeness of (iv) negation and disjunction (v) and of negation and conjuction follows from the definition of

$p \rightarrow q. \ .= \mathrm{Df}\ \overline{\mathrm{p}\vee}\ \mathrm{q}$
$p \rightarrow q. = \mathrm{Df}\ \overline{\ p}\mathrm{q}$

Independence may be proved in each case analogously.

Besides the above systems, the following can easily be verified to be the complete systems of independent connectives for propositional calculus :

(vi) system of implication and converse non-implication;

(vii) system of equivalence, disjunctions and '*f*'

(viii) non conjuction alone;

(ix) non disjunction alone.

6·7 Dualization. The process of dualization is commonly applied in mathematics as well as in a logistic system. If a theorem in mathematics or logic is proved, its dual can easily be deduced.

The dual $\overset{*}{\mathrm{a}}$ of a formula a is obtained by interchanging simultaneously the symbols 't' and 'f' whenever they occur, and each of the following pairs of operation : $\bigvee$ and-, $\alpha \rightarrow$ and$\leftarrow +$, $\leftarrow$and and $+\rightarrow$, $\rightarrow\leftarrow$and $\rightarrow+\rightarrow(\bigwedge)$, $\overline{\bigvee}$, 1 ; and the symbols 't, and 'f' where the operations of each pair are called duals of each other. It is to be noted that each of negation and conditioned disjunction is self-dual because each of the connective is left un changed by dualization. For example $a \rightarrow b. \rightarrow b$ and $a \leftarrow + b. \leftarrow + b$ are dual to each other so are $a \bigvee b$ and ab. The following theorems can easily be deduced.

THEOREMS 3. If b and c are duals of a, then $\vdash b \longleftrightarrow c$

(Any two duals of the same formula are equivalent)

4. If $\vdash a$ and $\overset{*}{a}$ is the dual of a. then $\vdash \overline{\overset{*}{a}}$ (Dual of a tautology is a contradiction).

5. If $\overset{*}{a}$ and $\overset{*}{b}$ are duals of a and b respectively and further if

(i) $\vdash a\rightarrow b$, then $\vdash .\ \overset{*}{b}\rightarrow\overset{*}{a}$

(ii) $\vdash .\ a\longleftrightarrow b$, then $\vdash \overset{*}{a}\longleftrightarrow\overset{*}{b}$

6.8 Canonical Form. Consider a formulation of the propositional calculus in which the primitive connectives are negation, conjunction and disjunction. Then a formula with conjunction, disjunction and negation over only atomic sentences is called in the 'Canonical form', or 'simplest form' or 'normal form'; for example $\bar{a}\vee b$ is the canonical form of the formula $a\rightarrow b$.

There are two kinds of canonical forms : disjunctive and conjunctive. The disjunctive form of a formula consists of a disjunction of conjunction, where each conjunct is negated or an unnegated atomic sentence. The conjunctive form of a formula consists of a conjunction of disjunction where each disjunct of every conjunction is either a negated or an un-negated elementary sentence. For example the formulas $a\,b\vee\bar{a}\,\bar{b}$ and $a\vee b\cdot\bar{a}\vee\bar{b}$ are respectively in disjunctive and conjunctive forms. The canonical forms are used to determine the tautological equivalence of two formulas. Consider $a\longleftrightarrow\bar{b}$ and $a\vee b\ .\ \overline{ab}$, which have the same canonical form as follows :

$$a\vee b\cdot\overline{ab}.\longleftrightarrow.a\vee b\,.\,\bar{a}\vee\bar{b}\,.\longleftrightarrow a\,\bar{a}\vee a\,\bar{b}\vee b\,\bar{a}\vee b\,\bar{b}$$

$$\longleftrightarrow.\ a\,\bar{b}\vee\bar{a}\,b.$$

Also $a\longleftrightarrow\overline{b}.\longleftrightarrow.a\,\overline{b}.\vee.\,\bar{a}\,b$

Thus the formulas have the same canonical form and are equivalent. The disjunctive form (conjunctive form) is used to recognise the tautological (contradictory) character of a formula if in the canonical expression there is at least two summands (factors) of the form $a\vee\bar{a}\ (a\ .\ a)$.

Canonical form of a formula in terms of conditioned disjunction, t and f is obtained by recursion with respect to the number of different variables which a formula contains. A formula containing no variable is in canonical form if and only if it is one of the two symbols t and f. A formula in which the distinct variables contained are, in alphabetic order, $q_1, q_2, \ldots\ldots, q_n$ is in canonical form if and only if it has the form $[P, q_n, R]$, where each of the formulas P and R contains all the variables $q_1, q_2, \ldots \ldots, q_{n-1}$ but does not contain q_n; and is in canonical form;

6.9 A system of strict implication. The implication '→' is a truth functional connective having the properties that a false proposition implies every proposition and if a proportion is true, then it is implied by every proportion".

Lewis [5] has introduced another implication relation '⇸' in such a way that $p ⇸ q$ has the meaning : q is deducible from p. The relation of strict implication has been introduced in terms of a new undefined concept, denoted by a diamond symbol □, standing for 'it is possible that' : $p ⇸ q .=.\ \mathrm{D^f}\Box^{(\mathrm{p}\ .\ \bar{q}),}$ $(p\ .\ \bar{q}$ is impossible). But this definition is the origin of the following paradoxes. If p is impossible, then $p \cdot \bar{q}$ is also impossible, so that $p ⇸ q$ is valid; leading to the paradox :

$\bar{\Box}\ \mathrm{p} ⇸ .\ p ⇸ q.$

If q is necessary, then $p\ .\ q$ is impossible and so in accordance with the definition $p ⇸ q$ is valid, leading to the paradox

$\bar{\Box}\ \bar{\mathrm{q}} ⇸ .\ p ⇸ q.$

Since the two paradoxes are provable in the Lewis and Langford systems of strict implication, it is impossible to interpret $p ⇸ q$ in that system as meaning "q is deducible from p".

In order to obtain a system free from these paradoxes, P. G. J. Vredenduin [6] has taken □, as an underfined idea, in addition to those of Lewis and Longford, the implication relation ⇸.

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5. C. I. Lewis, A survey of symbolic logic.

6. P. G. J. Vredenduin, A system of strict implication, J Symb. logic, 4(1939) 73--76

Instead of the above definition, he has only considered the relation $p ⥽ q . ⥽ \overline{\Box}(p . \overline{q})$, so that generally $\overline{\Box}(p . \overline{q}) ⥽ . p ⥽ q$ does not hold.

Seven new axioms are assumed, in addition to those of Lewis and Langford.

1. $pq ⥽ qp$.
2. $pq ⥽ p$.
3. $p ⥽ pp$.
4. $(pq)r ⥽ p : qr$.
5. $p ⥽ \overline{\overline{p}}$
6. $p ⥽ q . q ⥽ r . ⥽ . p ⥽ r$.
7. $p . p ⥽ q . ⥽ q$.
8. $\Box . pq ⥽ \Box p$.
9. The operation of substitutions.
10. If $p ⥽ q . q ⥽ p$ has been asserted, then p may be substituted for q
11. If p and q have been asserted, then pq may be asserted.
12. If p and $p ⥽ q$ have been asserted, then q may be asserted.
13. $\overline{p} ⥽ q . ⥽ . \overline{q} ⥽ p$.
14. $pq ⥽ r . ⥽ . p\overline{r} ⥽ \overline{q}$.
15. $p ⥽ q . ⥽ . pr ⥽ qr$.
16. $p ⥽ q . r ⥽ s . ⥽ . pr ⥽ qs$.
17. $p ⥽ \Box p$.
18. $p ⥽ q . \Box p . ⥽ \Box q$.
19. $p ⥽ q . ⥽ \overline{\Box}(p . \overline{q})$.

In order to avoid parodoxes, all the axioms of the system are altered to asserted propositions of the systems if the $\Box$ symbols are omitted. The axioms 8, 17, 18 and 19 are then altered to :

$pq ⥽ p,$

$p ⥽ p,$

$p ⥽ q \cdot p. ⥽ q,$

$p ⥽ q. ⥽ . \overline{\mathrm{p} \cdot \overline{q}}$

The first and third of the propositions are axioms of the system ; the second and fourth can easily be proved. It is trivial that if anyone of the four rules is applied to a formula or to formulas, containing $\Box$s and the $\Box$s are then dropped, the result will be the same as if tho $\Box$s had been dropped before the rule was applied. Then to every provable $\Box$ theorem, thus corresponds a provable theorem without $\Box$s.

It should therefore be proved that ;

$p ⥽ .\, q ⥽ p$ (Corresponding to the paradox) $\overline{\Box}\, \overline{p} ⥽ .\, q ⥽ p$

and

$\overline{p.} ⥽ .\, p ⥽ q$ (Corresponding to the paradox) $\overline{\Box}\, p ⥽ .\, p ⥽ q$

are not deducible from the axioms 1–7, 9–16. But, since $p ⥽ .\, \mathrm{q} ⥽ p$ is a consequence of $\overline{p} ⥽ .\, p ⥽ \mathrm{q}$, it is sufficient to prove that $p ⥽ .\, \mathrm{q} ⥽ p$ is not deducible from the axiom system. All these axioms are asserted propositions in the system of Lewis and Langford, so that $p ⥽ .\, \mathrm{q} ⥽ p$ is not deducible in the old system, it will not be deducible in the new system.

Consider the following metrices for $p\mathrm{q}$, $p ⥽ \mathrm{q}$, p and $\Box\, p$.

·	1	2	3	4	−
1	1	2	3	4	4
2	2	2	4	4	3
3	3	4	3	4	2
4	4	4	4	4	1

□	⥽	1	2	3	4
1	1	2	4	4	4
1	2	2	2	4	4
1	3	2	4	2	4
3	4	2	2	2	2

All asserted propositions on the system have one of the designated values 1 and 2.

Now choosing $p=2$ and $q=1$, $p ⥽ . q ⥽ p$ cannot be deduced from the axioms of the old system, for it has the value 4, and therefore $\overline{\Box}\, \bar{p} . ⥽ . q ⥽ p$ cannot be deduced from the axioms of the new system.

The proof remains valid if the axiom

$$\Box p ⥽ \overline{\Box} . \overline{\Box} p \text{ is added to the new system.}$$

The system of Vredenduin is a proper sub-system of the system of Lewis and Langford.

Exercise 6

A. EXAMPLES ON INDEPENDENCE OF AXIOMS

1. Let Σ_2 be a logistic system obtained by replacing the primitive 'o' by '$-$' and axiom 3 by the following

$3' : \overline{a} \to \overline{b} \to b \to a$ (Converse law of contraposition).

Establish an equivalence relation between Σ and Σ_2.

(Hint : by replacing $a \leftarrow + a$, i.e., $\overline{a \to a}$ by o, equivalence of Σ and Σ_2 can be established.

2. Establish the independence of the axioms of each of the logistic system Σ_3, Σ_4 and Σ_5 which have the same rules of inference as Σ have the primitives $\to$, $\longleftrightarrow$ and $-$ and have the axioms :

Σ_3 : formulas 1, 2′, 3 to 7 of exercise 2A

Σ_4 : formulas 1, 2″, 3 to 7 of exercise 2A

Σ_5 : formulas 1, 4 to 8 of exercise 2A.

(Hint the following matrix (designated value 1)

$\to$	1	2	3	$-$
1	1	2	3	3
2	1	1	2	2
3	1	1	1	1

$\longleftrightarrow$	1	2	3
1	1	2	3
2	2	1	2
3	3	2	1

will be found to satisfy all the axioms of Σ_5 but not the 8rth establishing its independence)

3. Prove the independence of axioms and rules of inference of the logistic system Σ_1 and also of Σ_2.

(Hint : a rule of inference *R* of a logistic system is called independent if, in the logistic system obtained by omitting *R* from other primitive rules, *R* is not a derived rule, or equivalently if there exists at least one theorem which can be proved without *R*. Except in the case of the rule of substitution, use matrix method.)

4. Let Σ_p be the logistic system of Principia Mathematica with substitution and modus ponens as the rules of inference and the following axioms :

(i) $p \vee p . \rightarrow . p$

(ii) $q . \rightarrow . p \vee q$

(iii) $p \vee q . \rightarrow . q \vee p$

(iv) $p \vee . q \vee r . \rightarrow . q \vee . p \vee r$

(v) $q \rightarrow r . \rightarrow : p \vee q . \rightarrow . p \vee r$

Are the axioms of Σ_p independent ?

(Hint : use matrix method to show that the fourth axiom is not independent.)

5. Let Σ_q be the logistic system obtained by deleting the fourth axiom of Σ_p. Discuss the independence of the axioms and the rules of Σ_q.

6. Let Σ_r be the axiom system obtained from Σ_p by deleting the third and replacing the second of Σ_p by the following 11′ $p . \rightarrow . p \vee q$. Discuss the independence of the axioms and rules of Σ_r.

7. Let Σ_s be the axiom system with the same rules of inference as Σ_p having the following axioms.

(i) $p \vee p . \rightarrow . p$

(ii) $p . \rightarrow . p \vee q$

(iii) $p \vee q . \rightarrow . q \vee p$

(iv) $p \to q. \to : r \vee p. \to r \vee q$

Discuss the independence and the rules of inference of Σ_s.

8. Let Σ_t be the axiom system with the same rules of inference as Σ_p and the following axioms.

(i) $p \vee p. \to .p$. (ii) $p. \to . p \vee q$

(iii) $q \to r. \to : p \vee q. \to . r \vee p$. Discuss the completeness of Σ_t.

B. EXAMPLES ON COMPLETENESS OF THE SYSTEMS OF CONNECTIVES

1. Taking non-conjunction as the only primitive connective, give definitions of the unary and remaining binary connectives.

(i) $\overline{ab}\ \overset{a/b}{.=.\ \mathrm{Df}}$ and $\overline{a} \overset{a/a}{=\mathrm{Df}=} a'$ say for brevity;

(ii) $a\,b \,.\longleftrightarrow.\, \overline{\overline{ab}} =_{\mathrm{Df}} a/b/.\,a/b\ (a/b)'$;

(iii) $a \vee b \,.\longleftrightarrow.\, \overline{\overline{a. \vee b.}} = \overline{\overline{a.\ b}} =_{\mathrm{Df}} a'/b'$;

(iv) $a \to b \,.\longleftrightarrow.\, \overline{a} \vee b := ._{\mathrm{Df}}\ a/b'$;

(v) $\overline{a \vee b} .=._{\mathrm{Df}} = (a'/b')'$;

Other operations can analogously be defined.

2. Taking non-disjunction as the only primitive connective, give definitions of the unary and remaining binary connectives.

(i) $\overline{a} =_{\mathrm{Df}} a \mathbin{\bar{\vee}} a = a^{\circ}$, for brevify ;

(ii) $a \vee b =_{\mathrm{Df}} \overline{a \mathbin{\bar{\vee}} b} =_{\mathrm{Df}} (a. \bar{\vee}. b)^{\circ}$;

(iii) $a\ .\ b =_{\mathrm{Df}} \bar{a} \mathbin{\bar{\vee}} \bar{b} =_{\mathrm{Df}} a^{\circ} \mathbin{\bar{\vee}} b^{\circ}$;

(iv) $a \to b =_{\mathrm{Df}} (a^{\circ} \mathbin{\bar{\vee}} b)^{\circ}$;

3. In terms of implication and negation, define the remaining operations. (Hint : Definitions analogous as with→and o)

4. Taking implication and converse implication as the only primitive connectives, define the remaining operations.

5. In terms of conditioned disjunction, *t* and *f*, define the remaining operations.

6. Show that equivalence, disjunction and *f* constitute a complete system of independent primitive connectives for the propositional calculus.

7. Establish the independence of the connectives in the following systems : (i) negation and disjunction, (ii) negation and conjunction ; (iii) implication and converse non-implication.

8. Taking conditioned-disjunction '*t*' and '*f*' as primitives, axiomatise the two valued propositional calculus.

(Vide A formalisation of two valued propositional calculus with self dual primitives, Math. Annalen 127(1954) 255—257).

9. Taking non-conjunction as the primitive, consider an axiomatisation of the propositional calculus.

C. EXAMPLES ON CANONICAL FORMS

1. Transform the following formulas into canonical forms :

(i) $p \leftarrow + \rightarrow q$ (ii) $p \longleftrightarrow . \, p \rightarrow q$ (iii) $p \vee q \, . \, \overline{pq}$.

(Hint : $p \leftarrow + \rightarrow q \, . \longleftrightarrow . \, p \longleftrightarrow \overline{q} \, . \longleftrightarrow . \, p \, \overline{q} \vee \overline{p} \, q$)

(ii) $p \longleftrightarrow . \, p \rightarrow q \longleftrightarrow p . \, (p \rightarrow q) . \vee . \, \overline{p .} \, \overline{p \rightarrow q}$

$\longleftrightarrow p . \, \overline{p} \vee q . \vee . \, \overline{p .} \, p \, \overline{q} \longleftrightarrow . p . \, \overline{p} \vee q \longleftrightarrow p \, \overline{p} \vee pq$

$\longleftrightarrow pq$

(iii) $p \vee q . \, \overline{pq} \longleftrightarrow p \vee q . \, \overline{p} \vee \overline{q} \longleftrightarrow p . \, \overline{p} \vee p \, \overline{q} \vee \overline{p} \, q \vee q \, \overline{q}$

$\longleftrightarrow p \, \overline{q} \vee \overline{p} \, q$

Alternative : $p \vee q . \, \overline{pq} \longleftrightarrow p \wedge q \longleftrightarrow p \leftarrow + \rightarrow q \longleftrightarrow p \, \overline{q} \vee \overline{p} \, q$)

2. Verify the following into the disjunctive normal forms.

Farmulas	disjunctive forms
(i) $p \vee \overline{r.}\ \ \bar{q}$	$\bar{p}\ q \ \vee\ \bar{r}\ \bar{q}$
(ii) $p . \longleftrightarrow . \overline{q \vee r}$	$p\ \bar{q}\ \bar{r} \ \vee\ \bar{p}\ q \vee \bar{p}\ r$
(iii) $\overline{\dot{p} \ \rightarrow . q \vee r :}\rightarrow . p$	$\bar{p}\ q \vee \bar{p}\ r$

3. Transform the following formulas into conjunctive normal Forms.

Formulas	conjunctive form
(i) $p \vee \overline{q.} \rightarrow r$	$p \vee r \cdot q \vee r$
(ii) $p \leftarrow + \rightarrow q$	$p \vee q . \bar{p} \vee \bar{q}$

7. Algebrisation of propositional Calculus.

It is possible to assign various algebraic structures to the propositional calculus : lattice, Boolean algebra, ring and Boolean ring.

Lattice. A partially ordered set L is said to be a lattice if every pair x, y of element in L has a supremum and a infimum, denoted by $x \vee y$ and $x \cdot y$ respectively, where $\cdot$ stands for the usual notation of meet The lattic has a unit (zero) if there exists an element 1 (0) such that $x \leqslant 1$ $(0 \leqslant x)$ for every x in L. The lattice L is called distributive if

$x . (y \bigvee z) = x\, y \bigvee x\, z$, for x, y, z in L ; and complemented if for every x in L, there exists an $\bar{x}$ in L such that

$x \bigvee \bar{x} = 1$ and $x\, \bar{x} = 0$

A logistic system, with negation, disjunction and implication can be assigned to a distributive lattice L with o and 1 such that for each element a of L, a greatest element $\bar{a}$ exists such that $a . \bar{a} = o$ (contradiction) and $a \bigvee \bar{a} = 1$ (logical necesssary proposition). Supremum and infimum are to be identified with disjunction and conjunction such that each is distributed over the other ; complementation, in the above sense with negation, and equality with mutual replaceability in any logical context (propositions having the same meaning structure are said to be logically identical and is denoted by '$=$').

The relation $\leqslant$ is called a lattice implication, i.e., $a \leqslant b$ whenever either of the relations $b = a \vee b$ or $a = ab$ holds in L. For if $b = a \vee b$, then $ab = a(a \vee b) = a$. Conversely if $ab = a$, then $b \vee ab = b \vee a = b$. It can easily be verfied that $\leqslant$ is reflexive and transitive and further if $a \leqslant b$ and $b \leqslant a$, then $a = b$.

Boolean algebra. A distributive and complemented lattice with unit and zero is called a Boolean algebra. The propositional lattice is a Boolean algebra. Generally a logistic system is a Boolen algebra with equivalence as the role of equality, and with disjunction and conjunction identified as the Boolean sum and Boolean product respectively. Consider the following example of Boolean algebra in a language L containing propositional calculus. The sentences of L fall into equivalence classes, such that two sentences belong to the same class if and only if they are logically equivelent in L (i.e. mutual consequences). The operations can at once be transferred to these equivalence classes. If s_1 and s_2 are sentences of L, and S_1, S_2 their eqnivalenceclasses, then $S_1 \vee S_2$ consists of the sentences equivalent to $s_1 \vee s_2$ and $\bar{S}$ consists of sentences equivalent to $\bar{s}$. Conjunction can analogously be defined. The set of equivalence classes in **L** is a Boolean algebra with respect to the operations of disjunction and negation thus defined for these classes.

Boolean ring. A Boolean algebra with unity is a Boolean ring, with multiplication and addition defined as :

$ab = a \,.\, b,\ a + b = a\,\bar{b}.\ \vee .\ \bar{a}\,b.$ or equivalently

$aa = a$ and $a + a = o$.

A logistic system is a Boolean ring :

(i) with non-equivalence, as the ring addition and conjunction as the ring multiplication.

(ii) with equivalence as the ring equality, nonequivalence as the ring addition and disjunction as the ring multiplication.

In each case a+b can be identified with $a \longleftarrow\!\!+\!\!\longrightarrow b$ as it is equivalent to $a\bar{b} \vee \bar{a}b$ such that $a \longleftarrow\!\!+\!\!\longrightarrow a$ the zero element. The verification of multiplication in each case is trivial.

It can easily be verified that a logistic system with the operations as in (i) is a commutative ring.

8. **Correlation with topology.** Tang Tsao-chen [7] began the study of the relation between the Lewis systems and topology and Mckinsey [8] has completed the establishment of a correspondence between one of the system of Lewis and topology. This correlation is very illuminating for logical calculus and topology.

A matrix is a system (K, D, $-$, $\star$, $\times$) where K is a set, D is a non-empty proper subset of K (the 'designated elements'), — and $\star$ are unary functions, and $\times$ a binary function, these functions being defined over K and class-closing on K. The operations —, $\star$, $\times$ correspond respectively to the primitive connectives negation, possibility, conjunction of Lewis. A normal matrix is one satisfying conditions corresponding to Lewis rules of inference adjunction and replacement. A "topological space in the wider sense" (S, $\star$) is defined by postulates taken from Kuratowski :

i. S is not an empty set.

ii. If α is any subset of S, then $\star\star\alpha$ is a subset of S.

iii. If α is any subset of S, then $\alpha \subset \star\alpha$.

iv. If α and β are any subsets of S then $\star(\alpha \vee \beta) = \star\alpha \vee \star\beta$.

v. If α is any subset of S then $\star\star\alpha = \star\alpha$.

vi. If 0 is the null set, then $\star 0 = 0$.

A topological expression A corresponds to that sentence of one of the systems of Lewis S_4 (in which $\Box\Box p = \Box p$) which results from A by replacing set variables by sentential variables, and the set operations by the sential connectives of Lewis :

$\star$, —, $\vee$, $\times$ $\rightarrow$, $\longleftrightarrow$ by the symbols $\Box$, —, $\vee$, $\cdot$, $\dashv$, $\equiv$.

Equations both of whose members are topological expressions are called topological equations. A topological formula is a topological equation whose right member is the symbol 1 and whose left member does not contain the constants 1 and 0. It has been shown that a topological formula $A=1$ is true for every topological space if and only if the corresponding sentence is provable in S_4 It has also been proved that the inclusion $A \subset B$ (or $A=B$) is true for every topolegical space if and only if $\alpha \dashv\!\!\!\rightarrow \beta$ (or $\alpha \equiv \beta$) is provable in S_4, where A and B correspond to α and β. From this it follows that Parry's results regarding the 14 distinct modalities of S_4 are equivalent to the results of Kuratowski. regarding the 14 distinct functions definable in topology by means of operations of complement and closure.

Stone [9] and Tarskai [10] have also established correspondence between the intuitionistic logic of Heyting and topology.

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[7] Tang Taso-chen, Algebraic Postulates and a geometric interpretations for the Lewis Calculus of strict implication, Bull. Am. Math. Soc. 44(1938) 737—744

[8] J. C. C. Mckinsey, A solution of the decision problem for the Lewis systems S_2 and S_4 with an application to topology, J. Symb. Logic, 6 (1941) 117—134.

[9] M. H. Stone, Topological representations of distributive lattices, casopis pro pestovani matematiky fysiky, 67(1937) 1—25.

[10] Alfred Tarskai, Der Aussagenkalkil and die Topologic, Fundamenta mathematicae, 31(1938) 103—134.

II. FUNCTIONAL CALCULUS

1. Different kinds of expressions. Our propositional Calculus is too meager to express any set of systematic facts, for, there are no means of symbolising common and proper nouns, pronouns, verbs, adjectives and adverbs in the propositional calculus. It is, therefore, necessary to introduce certain kinds of expressions corresponding to the many kinds distinguished in the grammer of the natural languages, so that we can translate sentences of everyday language into a new language consisting of new kinds of expressions and sentencial connectives. Such a device leads to an extension of the propositional calculus to the functional calculus.

1.1. Constants. The most familiar expression in the natural language is the proper name. A proper name denotes or names that of which it is a name. The relation between a proper name and what it denotes is called the name relation and the thing denoted the denotation. For instance, the proper name "Buddha" is said to denote or name the Indian philosopher, and he will be said to be the denotation of the name "Budha". Analogously "the author of the "Ram Charit Manas" denotes or names the Indian author, and he is the denotation both of this name and of the name "Saint Tulsidas".

According to the mathematical usasge, a proper name of a number is called a constant. This usage can be extended by removing the restrcition to numbers, so that the term constant becomes synonymous with proper name having a denotation.

1.2. Variables. A variable is a symbol whose meaning is like that of a proper name or constant except that the single denotation of the constant is replaced by the possibility of various values of the variable. The grammatical function of variables is analogous to that of pronouns and common nouns in the natural language which are denoted by the letters 'x', 'y', 'z',.. ...

Thus the sentence :

'Every thing is black or not black' is equivalent to the sentence : "For every x", x is black, or x is not black". Variables are free and bound as defined below in 1.7.

1.3. Functions. A function F is a correspondence which assigns to each element x of a given set X, a unique element (written F (x)) of some other set Y, where X is called the domain and Y the counter-domain (range).

The notion of correspondence in the definition can be replaced by the terms 'binary relation'. We may then define a function as a binary relation which relates to each element of its domain a unique element of the connter domain. For example, in a country, where polygamy is not practised, the relation of being a husband is a function, since, a person has atmost one wife at a given time. But the relation of being a mother is not a function, since one lady can be the mother of several children.

Remark 1.1. Since F is a many one correspondence associating with each element x of X, a unique element of Y, F is called a single valued function.

1.4. Forms. Expressions which are similarly obtained but in which the variables may, otherwise occur, provided that the expression has atleast one occurence of the variables as a free[1] variable. If a form contains a single free variable, we call a unary form, if just two variables binary; and so on. A form with exactly n different free variables is said to be an n-ary form. We give examples of forms from the domain of elementary mathematics. Let x and y are variables whose range is the set of real numbers. Then the forms x^x, xe^x, y, $\frac{1}{y}$ are unary, each having one free variable x or y. The forms ye^x, $\frac{x-y}{y}$ are binary each having x and y as free variables.

1.5. Propositional function. A proposition determines or is a concept of truth values. A variable whose range is the two truth values, i.e, a variable for which sentences (expressing proposition) be substituted is called a propositional variable. A form whose values are truth values and which, therefore, becomes a sentence when its free variables are replaced by apropriate constants is a propositional form. A function of a propositional form whose value consists exclusively of truth values is called a propositional function. A propositional function of n-variables is denoted by $F(x_1, x_2, \ldots x)$. A propositional function holds for a particular argument or ordered system of arguments, if its value for that argument or ordered system of arguments is true.

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[1] A variable is free if it is not a bound variable vide 1.7

1.6 Quantifiers. Consider the phrases :

"For all x, —"
"For every x, —" (1)

"For some x, —"
"For at least one x, —"
"There is an x such that —" (2)
"There exists an x such that —"

These are called quantifiers which act as operators. Each phrase in (1) is called a universal quantifier and is denoted by (x) $\forall_x$ or π_x. Every phrase in (2) is called an existential quantifier and is denoted by $\exists_x$ or Σ_x. There are other quantifiers known as formal implication $[\text{——} \longrightarrow_x \text{——}]$ standing for $\pi_x\, [\text{——} \longrightarrow \text{——}]$, formal equivalence $[\text{——} \longleftrightarrow x \text{——}]$ for $\pi_x\, [\text{——} \longleftrightarrow \text{——}]$, and formal non-conjunction $[\text{—}\ 1_x \text{——}]$ for $\pi_x\, [\text{——} 1 \text{——}]$, Ternary and n ary formal operations will be defined analogously; for example an n ary formal implication is denoted by $[\text{——} \longrightarrow x_1 . x_2 \ldots x_n \text{——}]$, standing for $\pi_{x_1\, x_2 \ldots x_n}\, [\text{——} \longrightarrow \text{——}]$.

1.7. Free and bound variables. An occurrence of a variable in a formula can be one of two types; either that occurrence is controlled by a quantifier, or it is not. The scope of a quantifier occurring in a formula is the quantifier together with the smallest formula immediately following the quantifier. For example, the scope of the existential quantifier in the following formula is indicated.

$$\underbrace{\Sigma_x\, \pi_y\, (xy=0)}_{\text{Scope}}, \quad \pi_x\, \pi_z\, (x+z=z+x).$$

An occurrence of a variable in a formula is bound if and only this occurrence is within the scope of a quantifier using the variable. In the above formula, every occurrence of variables is bound.

An occurrence of a variable is free if and only if this occurrence of variable is not bound.

For example in the formulas :

$\Sigma y\, (x>y)$, both occurrences of the variable 'y' are bound and the single occurrence of 'x' is free;

$\Sigma_x\, F(x)\ V\ G\,(x)$, the first two occurrences of 'x' are bound and the third occurrence is free.

A variable is a free veriable in a formula if and only if at least one occurrence of the variable is free, and a variable is a bound variable in a formula if and only if at least only one occurrence of the variable is bound.

Remark 1·7. A variable may be both free and bound in the same formula, but any given occurrence of the variable is either bound or free but not both.

Thus in a formula :

$x>0 \vee \Sigma_x(x<0)$,

the variable 'x' is both free and bound; the first occurrence is free, and the second and third occurrences are bound.

Exercise 1

1. Which of the following relations are functions ?

   (i) The relation of being a spouse.

   (ii) The relation of being a grandfather.

(iii) The relation of being a wife in a country where polyandry is not practised.

(vi) The relations of being a son.

2. Symbolise the following sentences :

   (i) All students are intelligent. $[\pi_x(Sx \rightarrow Ix)]$.

   (ii) Some students are intelligent $[\Sigma_x\ (Sx \cdot Ix)]$

   (iii) No students are intelligent $\pi_x\ [Sx \rightarrow \overline{Ix}]$

   (iv) Some students are not intelligent $[\Sigma_x Sx \cdot \overline{I}x]$

   (v) Some girls are pretty.

   (vi) All crows are black.

   (vii) There are black cows. $[\Sigma x\ Bx \cdot Cx]$

   (viii) All is well that ends well. $[Ex \underset{x}{\rightarrow} Wx]$

2. **Unary functional calculus (of one variable).** A convenient method of representing the values of the variables x in F(x) is by $x_1, x_2 \ldots x_n \ldots\ldots$, where $F(x_n)$ is a proposition. $F(x)$ gives the derivative propositions $F(x_n)$, $\pi_x\ F(x)$ and $\Sigma_x\ F(x)$ as defind below.

DEFINITION 2·1 $\Sigma F(x). =_D \Sigma_x(x). =\cdot_D F(x_1) + \ldots\ldots + F(x_n)\ .\ \ldots$

$$\pi F(x). =\cdot_D \pi_x F(x)\cdot =\cdot_D F(x_1) .\ F(x_2)\ F(x_3) \ldots F(x_n) \ldots\ldots\ldots$$

Remarks 2·1 Conjunction (·) and disjunction (+) are operations defined for only a finite number of terms. To extent these operations to an infinite number of terms requires new primitive terms π nnd Σ.

2·2 The formulas $\Sigma\ F\ (x)$ and $\pi\ F\ (x)$ are called existential statement and all statement (or a general statement) respectively.

Theorem 2·1 $\Sigma\ F\ (x) \longleftrightarrow \pi\ \overline{\overline{F(x)}}$, i.e, "it is true that some x is——"...equivalent to "it is false that all x is not——"

For $\Sigma x\ \ F(x) \cdot =\cdot_D\ \ F(x_1) + \ldots\ \ + F(x_n) + \ldots\ldots$

$$\longleftrightarrow \overline{\{F(\overline{x_1})\ \ F(\overline{x_s}) \ldots\ldots F(\overline{x_n})\}} \ldots\ldots$$

2·2.[2] $\pi\, Fx \longleftrightarrow \overline{\Sigma \overline{Fx}}$, ie, "it is true that all x is "—" is equivalent to "it is false that some x is not—" which can analogously be proved.

2·3. $\pi\ Fx \rightarrow \Sigma Fx$, i. e., "what ever is true of all is true of some" (subaltern).

For $Fx_1 .\ Fx_2 . Fx_3 \ldots\ldots Fx_n \cdots \rightarrow Fx_n$ and

$Fx_n \rightarrow Fx_1 + Fx_2 + \ldots\ldots + Fx_n + \ldots\ldots$

By the law of the transitivity of →,

$\pi\ Fx \rightarrow \Sigma\ Fx$.

$$2.4\ \ \substack{\Sigma\\ \pi}\left\{\begin{matrix} Fx \\ Fx \end{matrix}\right\}\begin{matrix} +\alpha \\ .\ \alpha \end{matrix}\ .\!\longleftrightarrow\!.\ \ \substack{\Sigma x\\ \pi x}\left\{\begin{matrix} Fx+\alpha \\ Fx.\ \alpha \end{matrix}\right\}$$

(Principle of importing into, or exporting out a proposition of the scope of Π or Σ, when the relation of the propesition to the other member of the expression is + or ·).

$$\text{For } \Sigma\ Fx + \alpha\ .=_D\ Fx_1 + \ldots\ldots + Fx_2 + \ldots\ldots + \alpha$$
$$.=.\ (Fx_1+\alpha) + (\ Fx_2+\alpha\ ) + \ldots\ldots$$
$$.=.\ \Sigma(Fx_1+\alpha).\text{Analogously for the rest.}$$

$$2.5\ \ \alpha \rightarrow \left\{\begin{matrix} \Sigma\ Fx \\ \pi Fx \end{matrix}\right\} \longleftrightarrow \substack{\Sigma x\\ \pi x}\left\{\ \alpha \rightarrow Fx\ \right\}$$

(principle of importing into, or exporting out a proposition of the seope of Π or Σ when the proposition is the implicans and the other expression in the implicate).

$$2\,6\ \ \substack{\Sigma\\ \pi}\left\{\ Fx\ \right\} \rightarrow \alpha \longleftrightarrow \substack{\pi_x\\ \Sigma_x}\left\{\ Fx \rightarrow \alpha\ \right\}$$

(Principle of impotation or expotation analogous to 2·5 when the proposition is in the implicate). The proof of each of the theorem follows readily from the following remark.

Remark 2·1. An operation in the implicate of an implication is not changed by being extended in scope to include the whole relation, but an operator in the implicans is changed from π to Σ and from Σ to π. This is due to the fact that $\alpha \rightarrow \beta \longleftrightarrow \overline{\alpha} + \beta$, where the sign of the implicans changes but the implicate remains the same; and to the princip le $\overline{\pi(\ )} \longleftrightarrow \Sigma(\overline{\ })$, $\overline{\Sigma(\ )} \longleftrightarrow \pi(\overline{\ })$

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[2] In unary function we may omit any parenthesis around the varaible

2·7. $\Sigma\, Fx + \Sigma Gx \longleftrightarrow \Sigma_x (Fx + Gx)$

$\pi\, Fx \cdot \pi Gx \longleftrightarrow \pi_x (Fx \cdot Gx)$

For each of + and · is associative and cammutative.

2·8. $\Sigma_x (Fx \cdot Gx) \rightarrow \Sigma Fx \cdot \Sigma Gx.$

$\pi\, Fx + \pi Gx \rightarrow \pi_x (Fx + Gx).$

For $\Sigma_x (Fx \cdot Gx) = (Fx_1 \cdot Gx_1) + \cdots + (Fx_n \cdot Gx_n) + \cdots,$

$(Fx_n \cdot Gx_n) \rightarrow Fx_n$ and $(Fx_n \cdot Gx_n \cdot) \rightarrow Gx_n.$

Hence $\Sigma_x (Fx \cdot Gx) \rightarrow \Sigma Fx$, and $\Sigma_x (Fx \cdot Gx) \rightarrow \Sigma Gx$

and so $\Sigma_x (Fx \cdot Gx) \rightarrow \Sigma F(x) \cdot \Sigma G(x)$

With two propositional funutions we can form two implication relations, $\Sigma_x (Fx \rightarrow Gx)$ and

$\Pi_x (Fx \rightarrow Gx)$. But $\Sigma_x (Fx \rightarrow Gx)$ states only that there is a value for which Fx is false or Gx is true. $\pi_x (Fx \rightarrow Gx)$ or $Fx \rightarrow_x Gx$ is the joint assertion of

$Fx_1 \rightarrow Gx_1,\ Fx_2 \rightarrow Gx_2,\quad Fx_n \rightarrow Gx_n \ldots\ldots$ where each assertion is a material implioation. It is thus a set of material implications and is called the formal implication whose properties are stated in the following theorems, which can easily be varified.

2 9,
$$\pi_x (Fx \rightarrow Gx) \longleftrightarrow \left\{ \begin{array}{l} \pi_x (\overline{Fx} + Gx) \\ \pi_x \overline{F(x).G\,x)} \end{array} \right\}$$

2·10. $(Fx \rightarrow_x Gx) \rightarrow (Fx_n \rightarrow Gx_n)$, i.e, if Fx formlly implies Gx, then Fx_n materialls implies Gx_n.

2·11. $Fx_n \cdot (Fx \rightarrow_x Gx) \rightarrow Gx_n$, i.e, if F is true of x_n and Fx formally implies Gx, then G is truc of x_n.

(One form of the sylloygism in Barbara)

2·12. $\pi_x(Fx \rightarrow Gx) \rightarrow \Sigma_x(Fx \rightarrow Gx)$

(follows from the principle of subaltern form)

2·13. $\pi_x \ (Fx \rightarrow Gx) \rightarrow \left\{ \begin{array}{l} \pi Fx \rightarrow \pi Gx \\ \Sigma Fx \rightarrow \Sigma Gx \end{array} \right\}$

2·14- $\pi_x \ (Fx \rightarrow Gx).\ \pi Fx \rightarrow \pi Gx$

(Modus ponens)

2·15. $\pi_x \ (Fx \rightarrow Gx)\ .\ \Sigma Fx \cdot \longrightarrow \cdot \Sigma Gx$, i e, if Fx alway implies Gx and Fx is sometimes true then Gx is also sometimes true.

2·16. $(Fx \rightarrow_x Gx)\ (Gx \rightarrow_x Hx) \rightarrow (Fx \rightarrow_x Hx)$

(Taransivity of formal implication, another form of the syllogism in Barabara)

2·17. $(Fx \rightarrow_x Gx) \cdot \rightarrow \cdot [\ (Gx \rightarrow_x Hx) \rightarrow (Fx \rightarrow_x Hx)\ ]$

2·18· $(Gx \rightarrow_x Hv) \cdot \rightarrow \cdot\ [\ (Fx \rightarrow_x Gx) \rightarrow (Fx \rightarrow_x Hx)\ ]$

2.19· $(Fx \rightarrow_x Gx) \longleftrightarrow \cdot \overline{G(x)} \rightarrow_x \overline{F(x)}$

Formal equivalence is a double formal implications, whose properties are stated in the following theorems.

2·20. $(Fx \longleftrightarrow_x G(x) \cdot \longleftrightarrow : (Fx \rightarrow_x Gx).\ (Gx \rightarrow_x Fx)$

2.21. $(Fx \longleftrightarrow_x Gx) \cdot (Gx \longleftrightarrow_x Hx) \rightarrow (Fx \longleftrightarrow_x Hx)$

2·22- $(Fx \longleftrightarrow_x Gx) \cdot \rightarrow [\ (Gx \longleftrightarrow_x Hx) \rightarrow \cdot (Fx \longleftrightarrow_x Hx)\ ]$

2·23. $(Gx \longleftrightarrow_x Hx) \cdot \rightarrow : [\ (Fx \longleftrightarrow_x Gx) \cdot \rightarrow (Fx \longleftrightarrow_x Hx)\ ]$

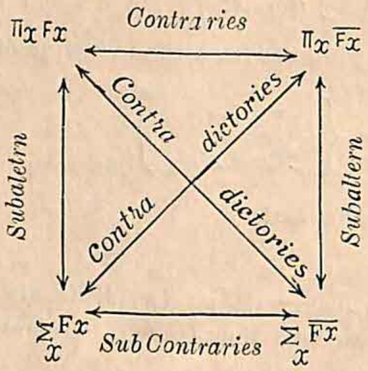
2·24. $(Fx \longleftrightarrow_x Gx) \cdot \rightarrow \cdot [\ (\Pi Fx \longleftrightarrow \Pi Gx)$; and

$(Fx \longleftrightarrow_x Gx) \rightarrow (\Sigma Fx \longleftrightarrow \Sigma Gx)$

2·25. $(Fx \longleftrightarrow_x Gx) \longleftrightarrow \left(\overline{Fx} \longleftrightarrow_x \bar{G}x \right)$

Exercise 2.

1. Formulate some of the forms of of the syllogism "Barbara" in the functional calculus of one variale. (vide theorms 2.11 & 2·16).
2. Construct the traditional square of oppositions in the functional calculus of one variable.



The term 'contradictory' expresses the negation of a statement, 'contrary' exclusion, and subalterm refers to the existential statement whereas subcontraries do not express exclusion.

3. Symbolise the following sentences both with all and existential operators.

   (i) No planet is self-luminous.

Hint: $(\pi_x (Px \rightarrow \overline{Sl(x)}) , \overline{\overline{\pi_x (Px \rightarrow \overline{Sl(x)})}} \longleftrightarrow \overline{\Sigma_x(Px.\overline{Sl(x)})}$

   (ii) There are no white crows.

   (iii) All girls are not pretty.

3 Binary and nary functional calculus. A propositional function $F(x,y)$ of two variables gives the derivative propositions $F(x_m, y_n)$, $\Sigma_y \pi_x F(x,y)$, $\pi_x \pi_y F(x,y)$ etc. The range of siganficance of $F(x,y)$ is a set of ordered pairs (x_m, y_n) such that eithere $F(x_m, y_n)$ is true or false. Since $F(x_m, y)$ is a propositional funtion of one veriable y, $\pi_y F(x_n,y)$, $\Sigma y\ F(x_n,y)$ etc are propositions, the meaning and proporties of which follow from tho preceeding Section, and $\pi_y F(x,y)$, $\Sigma_y F(x,y)$ $\pi_x F(x,y)$ and $\Sigma_x F(x,y)$ are pronositional functions of one variable.

The derivative propositions of $F(x, y)$ are defined below.

Definition 3.1

$$\left.\begin{array}{r}\pi_{x,\,y}\\ \Sigma_{x,\,y}\\ \\ \Sigma_x\ \pi_y\\ \pi_x\ \Sigma_y\end{array}\right\}\ F(x\ y)\ \underset{D}{\cdot\!=\!.}\ \begin{array}{r}\pi_x\\ \Sigma_x\\ \\ \Sigma_x\\ \pi_x\end{array}\left\{\begin{array}{ll}\pi_y & \\ \Sigma_y & \\ & F(x,y)\\ \pi_y & \\ \Sigma_y & \end{array}\right\}$$

Each of the above expressions can be expanded into a two dimensional array of propositions as follows :

$$\begin{aligned}\Sigma_x\ \pi_y\ F(x, y) &= \pi_y\ F(x_{1,y}) + \pi_y\ F(x_{2,y}) + \cdots\cdots\cdots + \pi_y\ F(x_{n,y}) + \cdots\\ &= [\ F(x_{1,y_1}) \cdot F(x_{1,y_2})\ \cdots\cdots\cdots\cdots\cdots\ F(x_{1,y_n})\cdots] +\\ &\quad [\ F(x_{n,y_1}) \cdot F(x_1 y_{,2}) \cdots\cdots\cdots\cdots\cdots F(x_{2,y_n})\ldots] +\\ &\quad [\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\ ] +\\ &\quad [\ (x_n, y_1) \cdot F(x_n, y_2) \cdots\cdots\cdots\cdots\cdots\cdots F(x_{n,y_n})] +\\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\end{aligned}$$

Remarks 3.1 $\Sigma_x\ \pi_y$ is rad, "For some x, every y, is such that—" and $\pi_y\ \Sigma_x$ is read, "For every y, some x is such that—".

Since $\Sigma_{x,y}\ F(x,y)$ and $\pi_{x,y}\ F(x,y)$ may be ragarded as involving only one variable (x, y), theorems analogous to the previous section at once follows. Other types of theorems are stated which can trivially he proved.

Theorem 3.1 or $\pi F(x)\ .\ \Sigma G(y)\ . \longleftrightarrow .\ \pi_x .\ \Sigma_y\ (Fx\ Gy))$

$$\Sigma\ Fx + \pi\ Gy \longleftrightarrow \Sigma_x\ \pi_y\ (\ Fx + Gy\ )$$

Function of three or more variables can be treated analogously. $F(x, y, z)$ may be treated as a function of three. Variables, or as a function of one variable, the ordred triple (x, y, z); analogous to the treatment of $G(x, y)$ as a function of x and y or of the ordred couple(x, y)

The derivative propositions of $F(x, y, z)$ are defined :

Definition 3.2 $\pi_{x,y,z}\ F(x,y,z) . =._{d}\ \pi_x\ \pi_{y,z}\ F(x,y,z)$

$$.=.\ \pi_y\ \pi_{x,z}\ F(x,y,z)$$

$$\begin{aligned}\Sigma_{x,y,z}\ F(x,y,z) &= .\Sigma_x \Sigma_{y,z}\ F(x,y,z)\\ &= \Sigma_y\ \Sigma_{x,z})\ F(x,y,z)\end{aligned}$$

Remark 3.2 $F(x, y, z)$ can be treated not only as a function of three variables or as a function of one, but also as a function of two, x and (y, z) or (x, y) and z or y and (x, z)

3.3 In general the convention of notation can be extended to functions of any number of variables and the analogues of the preceding theorems for functions of two variables will follow.

Exercise 3

A. Symbelise the following sentences :

1. All teachers love their students.

Hint : $\mathrm{T}(x, y).\ \mathrm{S}(y) \longrightarrow_{x, y} \mathrm{L}(x, y)$

2. A student reads a logic

Hint : $\Sigma_{x, y} [\mathrm{S}(x).\ \mathrm{L}(y).\ \mathrm{R}(x, y)]$

3. There are communists who do not admire Chouen Lai.

Hint : $\Sigma x[\mathrm{C}(x)\ \overline{\mathrm{A}(x, y_1)}]$

4, No wife hates her husband.

Hint : $[\mathrm{W}(x, y) \longrightarrow_{x, y} .\ \overline{\mathrm{H}(x, y)}]$

5. Radha gave Krishna a book.

Hint : $\Sigma x[\mathrm{B}(x).\ \mathrm{G}(y_1, z_1, x)]$

6. No clever boy marries a girl whom he does not like.

Hint : $\pi_{x, y} [\mathrm{C}(x).\ \mathrm{B}(x)\ \mathrm{G}(y)\ \mathrm{L}\overline{(x, y)} \rightarrow \mathrm{M}a\ (\overline{x, y})]$

B. Symbolise the sentences in A changing the operator π into Σ and vice-versa.

4. **Functions of Higher type.** In mathematics we are aquainted with the concept of a composite fanction, i.e., function of a - function. In the natural language we often speak of properties of properties and then make a function the argument of another denoted by F(G), where G is a function of the first type and F of the second type. The functional calculus of the second order is the extension of the functional calculus of the first order by applying quantifiers to sentential and functional variables. The derivative propositions of F (G) are :

$F(G_n)$, $\pi_G\ F(G)$, $\Sigma_G\ F(G)$, $\pi_G \pi_x F(Gx)$ etc.

Functional calculus of higher orders can be developed analogous to that of the first order.

Exercise 4

1. Symbolise the following sentences :

(i) Economy is a useful property $[G(F_1)]$.

(ii) Krishna has only useful properties.

$\pi_F\ [F(x_1) \rightarrow G(F)]$

(iii) There are useful qualities which are not developed by students.

$\Sigma_F\ [G(F)\ \pi x\ [B(x) \rightarrow \overline{D(F,x)}]$

(iv) Mahatma Gandhi had all the properties of a great man.

$\pi_F\ \pi_x\ [G_1(x) \rightarrow F(x)] \rightarrow F(x_1)$, where G_1 stands for great man, x_1 for Mahatma Gandhi.

2. Symbolise the principle of Mathematical Induction for natural numbers in the functional calculus of the second order,

$\pi_{x,y}\ \pi_P \{P(1),\ Suc(y, x) \rightarrow P(y)\} \rightarrow \pi_x P(x))$

Where suc (y, x) stands for y is the successor of x, i.e. if a property holds for the least number say, and if when it holds for any particular number, it also holds for its successor then the property holds for every natural number.

5. Axiomatization of the functional calculus of the first order. When the axioms of a system are infinite in number, they cannot be written out in full; and it is necessary to indicate them by one or more statements each introducing an infinite class of axioms. Formulation of a system based on an infinite number of axioms is represented by meams of an expression containing a form which has well formed formulas (w*ffs*)[3] as values. Every value of the expression is to be taken as an axiom. For conveniencc of statement, we shall indicate them by writing the expression itself in the same manner as an axiom.

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[3] A finite linear sequence of primitive symbols is called a formula, rules are given by which certain ones are designated as wellformed formulas. Certain among the weffs are laid down as axioms.

We shall now axiomatise the functional calculus of the first oder by means of five axim schemata. Let Γ be a system with :

(i) $\rightarrow$, $-$as connectives and Π as the universal operator;

(ii) $x, y, z, \ldots$ as individual valariables;

(iii) α, β, γ, ..as propositional variables;

(iv) F, G, H,... as functional variables;

(v) individual constants and functional contants;

(vi) modus ponens and generalisation *i.e.* from A, if a is an individual to infer π_a A, as the rules of inference.

5·1. Axima Schemata of Γ. The axiom Sehemata are as follows.

1. $A \rightarrow . B \rightarrow A$

2. $A \rightarrow . B \rightarrow C : \rightarrow : A \rightarrow B . \rightarrow . A \rightarrow C$

3. $\overline{A} \rightarrow \overline{B} . \rightarrow . B \rightarrow A$

4. $A \rightarrow_{a} B . \rightarrow . A \rightarrow_{\pi a} B$, where a is any individual variable which is not a free variable of A

5. ${}_{\pi a} A \rightarrow S_b^a A \mid$,where $S_b^a A \mid$ is the formula which results by suhstitutsion of b for each occurrence of a throughout A; a is an indi dual variable, b is an individual variable or constant, and no free occurrence of a in A is in a well formed part of A of the form π_b C

Remarks 5·1 One of the formulations of the proposonal Calculus is the hystem Γ°, which has modus ponens as the only rule of inference and the above first three axiom schemata.

5·2 According to 4 not every *wff* $A \rightarrow_{a} B . \rightarrow A \rightarrow_{\pi a}$ B is axiom but only those *wffs* of this form which satisfy the further condition that A contains no free occurrence of a.

5·3 The following examples will make the ideas clear as to the nature of 4 and 5.

One of the *ωffs* which is an axiom according to 4 is

$\alpha \rightarrow_{x} F(x). \rightarrow . \alpha \rightarrow \pi_{x} F(x)$. Again the w*ff* $G(x) \rightarrow_{y} H(y). \rightarrow . G(x) \rightarrow \pi_{y} H(y)$ is an axiom instance of 4; so is.

$F(x) \rightarrow_{y} [G(yz) \rightarrow_{z} H(z)] \rightarrow F(x) \rightarrow [G(yz) \rightarrow_{y,z} H(z)]$

and so on, an infinite number of axioms altogether.

The following *ωffs* which are axioms as instances of 5 are the following.

$\pi_{x} F(x) \rightarrow F(y)$

$\pi_{x} F(x) \rightarrow F(x)$

$F(x) \rightarrow_{x} \pi_{y} G(y). \rightarrow F(y) \rightarrow \pi_{y} G(y)$

$F(x,y) \rightarrow_{x} \pi_{z} G(x,z) \rightarrow F(y, y) \rightarrow \pi_{z} G(y,z)$

5·2 Definition schemata. In addition to the definition schemata 3—12 of Σ, we add the following further definition schemata.

DEFINITION 5·2·13 $\pi_{a} A =_{D} (a) A$

5·2.14 $\Sigma_{a} A =_{D} \overline{\pi_{a} \overline{A}}$

5·2·15 $[A \rightarrow_{a1 \cdots an} B] =_{D} \pi_{a1 \cdots an} \cdot A \rightarrow B$

5·2·16 $[A \longleftrightarrow_{a1 \cdots an} B] =_{D} \pi_{a1 \cdots an} \cdot A \longleftrightarrow B$

5·2·17 $A \mid_{a} B =_{D} \pi_{a} \cdot A \mid B$.

5·3 Theorem schemata of Γ. A theorem schema is an expression representing an infinite number of different theorems of a logistic system in the same way that an axiom schema represents many different axioms.

Since primitive symbols of Γ include propositional variables, every theorem of the propositional calculus in implication and negation as primitive connectives, is a theorem also of Γ. because the axioms of the system Γ° are included among the axioms of Γ and one rule of inference of Γ° is also a rule of inference of Γ.

The following important theorem schemata are stated.

5·3·1 $\vdash S^a{}_b\, A \mid .\to.\Sigma_a A$, with the same restrictions on *a* and *b* as in the axiom schemata 5.

Proof. By 5 $\vdash_{\pi a} \overline{A}\to \overline{S^a{}_b}\, A \mid$.

Hence by $\ulcorner^\bullet \vdash S^a{}_b\, A \mid \to \overline{\pi_a \overline{A}}\to\Sigma_a A$.

5·3.2 $\vdash \pi_a A\to\Sigma_a A$.

Proof. By 4 $\vdash \pi_a A\to A$.

By 5·3·1 $A\to\Sigma_a A$.

By the transitive law of implication $\vdash \pi_a A\to A.\to A\to\Sigma_a A$, the result follows :

5·4 **Consistency**. From a *wff* of $\ulcorner$, we obtain an associated formula of the propositional calculus denoted by *wff*° by first forming the associated quantifier free formula and then, in the latter, replacing every well formed part $F(x_1,\ldots x_n)$ by a propositional variable not previously occurring, in accordance with the following rule : two well formed parts $F(x_1,\ldots x_n)$ and $G(y_1\ldots y_m)$ are replaced by the same propositional variable if and only if F and G are the same functional variable or functional constants which can happen only when $m=n$.

For example, the *wffs* $F(x)\to_y G(y)\to F(x)\to\pi_y G(y)$ and $F(x,y)\to_x \pi_z G(x,z)\to.\ F(y,y)\to \pi_z G(y,z)$ have the same *wff*° as $\alpha\to\beta\to.\ \alpha\to\beta$.

Clearly all the *wffs*° of a given *wff* of $\urcorner$ are variants of one another. Hence if one of the *wffs*° of a given *wff* of $\ulcorner$ is a tautology, so are others. It can easily be verified that every *wff*° of an axiom of $\urcorner$ is a tautology and every theorem of $\ulcorner$ has a tautology as *wff*°.

Cleary if a *wff* A has a tautology a *wff*°, its negation $\overline{A}$ has as *wff*°, not a tautology but a contradiction. Hence not both A and $\overline{A}$ can be theorems. Thus $\ulcorner$ is consistent with respect to the transformation of A inta $\overline{A}$ and absolutely also.

5·5 **Incompleteness** of $\ulcorner$. Since the axioms all have the property of having a tautology as *wff*°, the addition to $\ulcorner$ of ano ther axiom having a tantology as wff° would not alter the property of the system that every theorem has a tantology as wff° and therefore would not destroy the consistency of the system.

It follows that Γ is incomplete with respect to the transformation of A into $\overline{A}$ and also absolutely.

EXERCISE 5.

1. Let Γ^1 be a system with :
   - (i) negation, conjunction and disjunction as its primitive connectives;
   - (ii) universal and existential quantifiers;
   - (iii) generalisation and modus ponens (in the form AUB and A to infer B);
   - (iv) individual propositional and functional variables and constants ;

Give suitable axiom schemata and establish the equivalence of Γ and Γ^1 in an appropriate sense.

2. Let Γ_1 be the system whose primitive symbols include all the propositional variables and all the functional variables but no individual and no functional constants; whose connectives and quatifiers are the same as of Γ and whose axioms are the five following :

(1) All the three axioms of the system Σ_1

(4) $\alpha \rightarrow_x Fx \rightarrow . \ \alpha \rightarrow \pi_x F_x$

(5) $\pi_x F(x) \rightarrow F(y)$.

What are the rules of inference of Γ_1 ?

(Modus ponens, Generalisation, rule.of alphabetical change of bound veriables, rule of substitution for individual variables, rule of substitution for propositional variables, rule of substitution for functional variables)

(3) Distnguish what are the special functional calculi of Γ ? Axiomatise each such special calculus.

(i) Γ_1 pure functional calculus of the first order

(ii) Γ_{11} pure unary functional calculus of the first order

(iii) Γ_{12} pure binary functional calculus of the first order

(iv) Γ_{1n} pure nary functional calculus of the first order

6. Normal forms and dualization. Analogous to the propositional normal forms and duals certain normal forms and a dual of a formula also exist in the functioned calculus.

6.1 **Normal form**. A formula may be replaced by one in which all quantifiers occur unnegated at the begining and are not separated by parentheses, so that their scopes all extend to the end of the formula. This normal form is called prenex normal form. The transformation into the normal form is effected as follows :

Replace $\rightarrow_z$, $\longleftrightarrow_x$ and 1_x in the formulas by their meanings. By repeated rule of transformation of π into Σ and $\sum$ in to π_x we can easily arrive at a formula in which the negations bar stands only over the proposition and functional variables. We now rewrite the bound variables in such a way that all quantifiers have different variables. Thus instead of π_x $F(x) U \pi_x G(x)$, we write π_x F $(x) U \pi_y G(y)$. From the logical expression thus obtained, we now get the normal form by replacing all the quantifiers at the begining of the formula in the order in which they occur and leaving everything alse unaltered. The scopes of the quantifiers is thus extended to the end of the formula.

A formula in prenex normal form is in Skolem normal form if it has no free individual variables and its prefix contains at least one existential quantifier and every existential quantifier in the prefix precedes every universal quantifier i.e., the prefix is of the form $\Sigma x_1, x_2 \ldots x_n \, \pi_{y_1}, y_2 \ldots y_n$

6·2 **Dualization**. The dual of a formula of Γ is obtained by interchanging simaltaneously π and Σ and the connectives as in system Σ.

Exercise 6

1. Find the prenex normal form of each of the following.

(i) $\pi y\, F(x, y) \rightarrow_{\pi y} F(x,x)$

(ii) $F(x) + \overline{\pi_y F(y)}$

(iii) $\Sigma_x G(x,y,z) \longleftrightarrow \pi y\, F(x,y,z)$

2. Find the Skolem normal form of each of the following :

(i) $\pi_x G(x) \rightarrow \Sigma_x G(x)$

(ii) $\alpha \rightarrow \Sigma_x \alpha$

(iii) $F(x) \rightarrow_x G(x) \rightarrow \Sigma_x F(x) \rightarrow \Sigma_x G(x)$

3 Give the duals of the axiom schemates 1, 3, 4 and 5.

Axim schemates	Duals as theorems schema
(1)	$\vdash B \leftarrow\!\!+ A \rightarrow A$
(3)	$B \leftarrow\!\!+ A \rightarrow . \bar{A} \leftarrow + \overline{B}$
(4)	$A \leftarrow + \Sigma_a B \rightarrow \Sigma_a [A \leftarrow\!\!+ B]$
(5)	if a is not free in A

4. Prove that the duals of the following theorem schemata are in the following table.

Theorem Schemates	Duals
(i) $\pi_{a,b} A \longleftrightarrow_{\pi b,a} A$	$\vdash \Sigma_{a\cdot b} A \longleftrightarrow \Sigma_{b,a} A$
(ii) $\vdash_{\pi a} A \longleftrightarrow A$, if a is not free in A	$\vdash \Sigma_a A \longleftrightarrow A$, if a is not frce inA
(iii) $\vdash \Sigma_a \overline{A} \longleftrightarrow \pi_a \overline{A},$	$\overline{\vdash \pi_a} A \longleftrightarrow \overline{\Sigma_a} A$
(iv) $\vdash \pi_a A \longleftrightarrow \pi_b B,$ if there is no free occurrence of b in A and no free occurrence of a in A in a well formed part of A of the form $\pi_b (C)$ and B is $S^a{}_b A$	$\vdash \Sigma_a A \longleftrightarrow \Sigma_b B,$ with the same restrictions on a and b

4. What is the advantage of the prenex normal form of a formula ?

(The aduantage of the prenex normal form consists in the fact that the expressions after the quantifiers called matrix may be treated just like a sentential Combinatins)

5. Write short notes on prenex and Skolem normal forms of a wf.

(Hint. A wf $(Q_1 x_1)(Q_2 x_2)......(Qnx_n)$ A when each $(Q_i x_i)$ where each Q_i is a universal or existential quantifier, $xi \neq x_i$ for $i \neq j$ and A contains no quantifiers is said to be in prenex normal form. The can $n = o$ i.e. when there are no quantifiers at all may also be included

It can be proved that for each wf A, it is possible to construct an equivalent wf B in prenex normal form, i.e , their is an effective procedure for transforming any wf A in to a wf B in a prenex normal form such that $\vdash (A \longleftrightarrow B)$. For a pure first order functional calcules (having no individual and functional constants), it is possible to find a very simple prenex normal form. A wf in prenex normal form such that all existential quantifiers precede all universal quantifiers is said to be in Skolem normal form. In a pure first order functional calculus, there is an effective process assigning to each wf A another wf B in skolem normal form. It is to be noted that a Skolem normal form for A is not necessarily logically equivalent to A, in contradiction to the prenex normal form.

7 Validity and satisfiability.

A wf true under all interpretations is said to be valid α. A wf is said to be satisfiable iff it is true-under some interpretations. Clearly a wf A is valid (resp. satisfiable) iff $\overline{A}$ is not satisfiable (resp. valid). A is contradictory īff A is valid, or equivalently, iff A is false for every interpretation. A logically emplies B iff $A \rightarrow B$ is valid. A and B are logically equivalent iff $A \longleftrightarrow B$ is logically valid.

We shall only state (without their proofs) three principal metamathematical theorems concerning first order functional calcules FOFC

7·1 G̈odel completness of FOFC. FOFC is axiomatizable in the sene that oxioms $\{Ax\}$ and rules $\{R\}$ can be found such that a formula of FOFC is deducible from $\{Ax\}$ by means of $\{R\}$ iff i t is valid.

7·2 Church's theorem. There is no effective algorithm for determining whether or not formulas of FOFC are valid.

7·3 Löwen heim's property. A formula of FOFC is valid iff it is true for all interpretations in some countably infinite domain.

The study of the elementary system FOFC derives considerable significance from the fact that any instance of the general deducitility can be effectively reduced to the question of whether or not a particular formula of FOFC is valid

An effective algorithm can be given such that for any axioms $\{Ax\}$, rules $\{R\}$ and statement T, we can find by this algorithm a formula A of FOFC with the property that T is deducible from $\{Ax\}$ by $\{R\}$ iff A is valid over a countably infinite domain; or equivalenlly, if we take B to be $\overline{A}$, then there is an effective algorithm for finding a formula B of FOFC such that T is deducible from $\{Ax\}$ by $\{R\}$ iff B is not satisfiable over a countable infinite domain. The problem of validity in FOFC can be further reduced within it to the problem of validity for certain subclass of formulas, Such reduction theorem may take the form :

If α the class of formulas of FOFC, β is a specified subclass ef α, then there is an effective transformation (algoritism) $f : \alpha \rightarrow \beta$ such that for any A, A is valid iff fA is valid.

Exercise 7

1. If A does not contain x free, then $\pi_x[A\rightarrow B]\rightarrow[A\rightarrow\pi_x B]$ is valid
(Hint : true for all interpretations)

2 If t is free for x in A, then $\pi_x A(x)\rightarrow A(t)$ is valid

3 The following wfs are logically valid :

(i) $\pi_x A\rightarrow\sum_x A$;
(ii) $\pi_{x,y} A\longleftrightarrow\pi_{y,x} A$;
(iii) $\Sigma_x\pi_y A\rightarrow\pi_y\sum_x A$.

4. For a given interpretation, a wf without free variables is called a closed wf which represents a proposition which is true or false, whereas a wf with free variables stands for a relation on the domain of the interpretation which may be satisfied for some values in the domain of the free variable and not satisfied for the others.

If A is a closed formula, show that A logically emplies B iff B is true in every interpratation in which A is true; but this is not always the case when A has free variables

5. If the free variables of A are $y_1, y_2, \ldots y_n$, then A is satisfiable iff $\sum_{y_1, y_2, y_n}$ A is satisfiable.

6. Introducing appropriate abbreviatons, write the sentences of the following arguments as wfs, and determine whether the conclusion is logically implied by the conjnction of the premisses.

(a) If every ancestor of an ancestor of an individual is also an ancestor of the same individual, and no individual is his own ancestor, then there must be a person who has no ancestor.

(b) Any barber in a vallage shaves exaclty those who do not shave themselvs. Hence there is no barber in the village.

7 If for some cardonal $\alpha \geqslant w$, a wf A is true in every interprelations of cardinality α, then A is logically valid.

8. If a wf A is true in all interpretation of cardinality α, then A is true in all interpretations of cardinality $\leqslant \alpha$.

9. For a given wf A, there are only a finite number of interpretations of A on a given domain of finite cardinality.

10. A closed prenex wf $\Sigma_{x_1, x_2 \cdots, x_n} A$, where A contains no quantifiers and individual constants, is valid iff it is true for all domains of one element.

11. A closed prenex wf $\pi_{x_1, x_2 - x_n} \Sigma_{y_1, y_2, y_m} A$ with $m \geqslant O$, $n > o$, is valid iff it is true in every interpretation with domain of n objects (A is assumed to have no quantifiers and individual constants)

12. A wf is called k—valia if it is true in all interprelations having k elements. Then k + 1 validity implies k—validity.

III THE CALCULUS OF CIASSES

3·1 Class and membership. A unary propositional function is satisfied by an argument, if its value for that argumenr is truth. A Class can therefore, be identified with a unary proposinal function and the membership in the Class with satisfaction of the unary propsorstinal funclion. The relation of an individual member of a class tothe class itself is symtsolised by ϵ (resembling to the Greek letter epsilon)

3·2 Operations with classes. The application of propositional operations leads to the definition of operations in Classes

DEFINIFION of 3·2·1 $x \epsilon \overline{F} =_{D} \overline{x \epsilon F} = x \bar{\epsilon} F$

(Comprlementary Class)

$$x \epsilon F + G =_{D} (x \epsilon F) U (x \epsilon G)$$

(disjunct or union class)

$$x \in F\,G \underset{D}{=} (x \in F \;.\, (x \in G)$$

(Conjunct Class or Intersection Class)

$$x \in F \rightarrow G = (x \in F) \rightarrow (x \in G)$$

(Implication Class)

$$x \in F \longleftrightarrow G \underset{D}{=} (x \in F). \longleftrightarrow (x \in G)$$

(Equivalence Class)

$$(F=G) \underset{D}{=} x \in F. \longleftrightarrow_{x}.\; x \in G$$

(Identity of Clasesses)

The identity of Clasess is defined in terms of an equivalence of propositions.

The all oporatr is exprered by the use of the universal Class V (domain of individuals of discourse).

Thus the formuas $\pi_x\, [\, F\,(x) + F\,\overline{(x)}\,]$ and $\pi_x\, F\,(x).\, \overline{F\,(x)}$ can respectively be written $F+\overline{F}=V$ and $\overline{F.\,F}=V$ or more simply

$$\overline{F.\,F}=\wedge$$ (null Class, a class having no member)

The existential opeator can be exprersd by the negation of a statement saying that a class is identical with the null class. Thus $F \neq \wedge$

means $\sum_x F(x$l (nonempty Class).

If an implication class is identical with the universal Class i,e, $F \rightarrow G = V$, thes

$x \in F \rightarrow_x x \in G$. In this Case

F is said to be included in G and is denoted by an inclusion relation $F \subset G$.

3.3 The Scope of the calculus of classes : The scope of the calculus of classes can be extended in symbolizing some of the sentences of everyday language and in defining certain structures which are attributes to classes.

3.3.1 Symbolizing sentences. Consider statements of the form (i) 'Some——are——', (ii) 'No——are——' (iii) 'Some—— are not——' (iv) 'All——are'—'If the blanks in one are filled by common nouns or the second blank is filled by an adjective, then the statement means that there exists something which is described by both terms i.e., the intersection of the two corresponding classes is not empty. Thus, for instance :

Some Indians are philosphers means that there exists at least one person who is both an Indian and a philospher and is accordingly symbolised :

$IP \neq \wedge$, where I and P respectively stand for the class of Indians and the class of philosphers. If the blank in (ii) are filled by common nouns, then the statement means that nothing belongs both to the class corresponding to the first noun, and to the class corresponding to the second noun, i.e., intersection of these two classes is impty. For instance :

'No Indians are philosphers' is translated :

$$I.P = \wedge$$

If the blanks in (iii) are filled by common nouns, then the statement means that there exists something which belongs to the class corresponding to the first noun, and do not belong to the class corresponding to the second class : i.e., the intersection of the first class with the complement of the second is not empty.

Thus the sentence :

$$I.\overline{P} \neq \wedge$$

'Some Indians are not philosphers' is translated:

When the two blanks in (iv) are filled with common nouns such as 'Indians' and 'Philosphers', or the second blank is filled by an adjective such as mortal, the corresponding statement means that the class of elements described by the first noun is a sub class of the class of elements described by the second noun. Thus for examble, the sentences.

All Indians are philosphers. All Indians are mortal : are symbolised.

$I \lhd P$, or $I+P=P$, or $I \cdot \overline{P}=\wedge$

$I \lhd M$, where M stands for the class of mortal beings.

3,3.2 Altributes to classes. Numbers are altributes to classes (sets). A relation $\lhd$ in a given set X partialy orders X, or is a partial ordering in X if it is both

(ii) transitive, i. e. whenever x y and z are in X and $x \lhd y$ and $y \lhd z$, then $x \lhd z$;

(ii) antisymmetric, i. e., whenever x and y are in P and $x \lhd y$ and $y \lhd x$, then $x=y$.

A number is defined as a non empty partially ordered set [i]. In otherwords, a number X is a set of elements x, y, z,......... connected by a transitive and antisymmetric relation $x \lhd y$ between certain pairs of elements. Isomorphic partially ordered sets represent equal numbers while the concept of inequality between two numbers is translated in to the concept of sub number.

The usual concept of cardinal numbers and ordinal numbers corresponds to the extreme cases of a totally un-ordered set and a totally ordered set.

3.3.2. A cardinal number X means a number X such that $x \lhd y$ in X imblics $x=y$. In other words, an unordered set X in which $\lhd$means$=$, often called a common ordered set $(X, =)$ is called a cardinal number X.

With any number X, may be associated its cardinal number No 'X, this is composed of the elements of X, but with a new inclusion relation, which is allowed to subsit only between identical elements.

An ordinal number No 'X is a number X, every nonempty subset S of which has a greatest (first) element s satisfying $x \lhd s$ for all $x \in S$. Clearly this is the usual well ordering condition.

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(1) Birkhoff Generalised Arthmetic, Duke Jour. of Math. 1942

Latlice is a number X, in which any two elements x and y have a unique supremum $x+y$ $(x \vee y)$ and a unique infimum $x.y$. Chain is a number X such that for any x and y in X, either $x \subset y$ or $y \subset x$. Thus any ordinal number is a chain.

Bounded number means a number having a least element O and a greatest element 1 such that $O \subset x \subset 1$ for all $x \in X$

A natural number is a finite ordinal number.

EXERCISE 3

1. Symbolise the following sentences in the calculus of classes :

(i) Women are fickle $(W \subset F)$

(ii) Men are numerous $(M \in N)$

(iii) All soldiers are clean and hardy $S \subset C.H$

(iv) Freshmen are ignorant but enthusiastic $F \subset I.E$

(v) Pleders and doctors are liars. $P+D \subset L$

For $P \subset L$ and $D \subset L$ implies $P+D \subset L$

(vi) Some Indians drink both wine and milk

$I. W. M. \neq \Lambda$

(vii) Some Indians who drink tea do not drink either coffee or milk.

(IT). $(C+M) \neq \Lambda$, for the general form of the statement is ; some 'S are not P' and the subject and predicate respectively are

AT and C+M.

(viii) A Philospher drinks neither tea nor coffee.

(ix) No Frenchman is an Indian.

(x) All wine drinkers drink either tea or coffee.

2. What is arithmetico logical operation and relation ? (cardinal numbers are attributes to classes. If the two classes are disjoint (having no common elements), the logical sum of the two classes is arithmetical sum of the corresponding cardinal numbers. But when they are not disjoint the arithmetical sum of two classes is the sum of the cardinal numbers of the union and the intersection of the classes.

The inclusion relation between the two classes entails an ordering relation between the corresponding cardinal numbers.

IV. RELATIONAL CALCULUS

A relation may be identified with a binary propostional function and the relation is said to hold between an ordered pair of elements, or the elements are said to stand in that relation, or to bear that relation one to other if and only if the binary proportional function is satisfied by the ordered pair.

In the binary function R (x, y), the argument x is called the refrent, the argument y the relatum. The class of all refrents is called the demain, the class of all relata the converse domain and domain and converse domain taken together constitute the field of the relation. For example, if paternity be the relation father form its domain, children its converse domain and father and children taken together form its field.

4·1. Axiomatization.

Tarski has given two methods of setting up the foundations of the calculus of relations : One making use of the functional calculus, the other using only propositional calculus.

4·2. Tarski's first method.

In the first method, the undefined symbols specific to the functional calculus are supplemented by (i) a special class of variables called "relation variables" (ii) a set of four relation constants (O, 1, O', 1' standing for null, universal, diversity and identity relation respectively) (iii) a set of six oprations complement, logical addition and multiplecation. (called Boollcan operations) converse, relative addition (+'), relative and produce (·[1])

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1. Tarski Alfred, calculus of Relations, Jour. Sym. Logic (1941) 73—89
2. Mckinsey J. C. C., Postulatcs For the calculus of relation Jour Sym. Logic (1940) 85—97

multiplication, called relative or Peircean operations). (iv) the rules of inference being substitution, modus ponens and the rules concerning the use of quantifiers.

The axioms are the following :

1. $\pi_{x,y}\ x1y.$
2. $\pi_{x,y}\ \overline{xoy}$
3 $\pi_{x}\ x1'x$ (identity relation between individuals)
4. $\pi_{x,y,z}\ [R\,(x, y).\ y\,1'\,z \rightarrow R\,(x, z)]$
5. $\pi_{x,y}\ [O'\,(x, y) \longleftrightarrow .\ 1'\,\overline{(x, y)}]$
6. $\pi_{x,y}\ [\overline{R}\,(x, y) . \longleftrightarrow .\ R\,\overline{(x, y)}]$ (Complement)
7. $\pi_{x,y}\ [\breve{R}\,(x, y) . \longleftrightarrow .\ R\,(y, x)]$ (Converse)
8. $\pi_{x,y}\ [R+S\,(x+y) \longleftrightarrow R\,(x\ y)+S\,(x, y)]$
9. $\pi_{x,y}\ [R+'S\,(x, y) . \longleftrightarrow .\ \pi_{z}\ (R(x, z)+S(z, y)]$
10. $\pi_{x,y}\ [R \cdot S\,(x, y) . \longleftrightarrow .\ R\,(x, y).\ S\,(x, y)]$
11. $\pi_{x,y}\ [R \cdot' S\,(x, y) . \longleftrightarrow . \Sigma_{z}\ (R(x, z).\ S\,(z, y)]$
12. $R=S . \longleftrightarrow .\ \pi_{x,y}\ (R(x, y) \longleftrightarrow S\,(x, y))$ (Identity between Relations)

With this set of axioms and with a slight modification of the substitution rule (the replacement of relation variables not only by other relations but also by any relation designations, which are constructed from relation variables, relation constants and operation signs) an elementary theory of the binay—relations is thus obtained.

To obtain the calculus of relations, it is only necessary to confine one's attention to these propositions and theorems which contain no individual variables.

The following fifteen theorems are stated :

(i) $R=S.\ R=T : \rightarrow . S=T$
(ii) $R=S. \rightarrow : R+T=S+T.\ R.T=S.T$
(iii) $R+S=S+R.\ R.S=S.R$

(iv) $(R+S).T=(R.T)+(S.T).\ (R.S)+T=(R+T).(S+T)$

(v) $R+0=R.\ R.1=R.$

(vi) $R+\bar{R}=1.\ R.\bar{R}=0$

(vii) $T=0,$

(viii) $R^{\smile\smile}=R,$

(ix) $\overset{\smile}{R\,;S}=S^{\smile}\,;R^{\smile}$

(x) $R.'(S.'T)=(R.'S).'T$

(xi) $R.'1'=R$

(xii) $R.'1=1+1.'\bar{R}=1$

(xiii) $(R.'S).T^{\smile}=0 \rightarrow (S.'T).R^{\smile}.=0$

(xiv) $0'=\overline{1'}$

(xv) $R+'S=\overline{\bar{R}\,.'\,\bar{S}}$

By transforming the theorems on the basis of the twelve axioms into equivalent sentences which contain no specific constants of the calculus of relations, the resulting sentences can be proved by the usual methods of the functional calculus. For example from theorem (xii), we obtain a simple theorem of functional calculus by such transformation, $\pi_x\,\Sigma_y\,R(x,y)+\pi_y\Sigma_x\,\overline{R(x,y)}$, which is more familiar in the following form :

$$\Sigma_y\,\pi_x\,R(x,y)\rightarrow\pi_x\,\Sigma_y\,R(x,y)$$

4.1.2 **Tarski's second method.** Since we are interested only in the calculus but not the theory of relations, the above method is not suitable. The second method deletes the individual variables and quantifers from the original theory then, with a new definition of what constitutes meaningful proposition, certain propositions can be singled out as the axioms of the new theory. These axioms fall naturally into two classes, the first class comprising any set of axioms for the propositional calculus, and the second class consisting of the set of fifteen theorems obtained above. The formal proofs of the theorems of the calculus of relations are then obtained using only the rules of inference of the propositional calculus, and gain considerably both in brevity and in elegance.

4.1.3.13 **Mokinsey's Postulates** : Mcknisey has presented a set of seven postulates for the calculus of binary relations. The undefined notions are $\triangle$, $\angle$ and $\cdot'$, interpreted as a set of binary relations, inclusion and relative product respectively. 'Atom', 'complete atomic Boolean algebra' and 1, (Universal element) are defined; thus P is atom if $P \neq O$ (O, null Boolean element) and if the only values of X for which $X \angle P$ are O and P; a Boolean algebra is a complete atomic one if every element except O contains an atom and if the sum and product of every set of elements exist. Let $(\triangle, \angle, \cdot')$ be a system in which $\angle$ and $\cdot'$ satisfies the following properties, called postulates.

P1. $\triangle$ is a complete atomic algebra with respect to $\angle$.

P2· $\triangle$ is closed with respect to $\cdot'$

P3. .' is anociative.

P4. If $X \angle Y$ and $U \angle V$, then $X \cdot' U \angle Y \cdot' V$.

P5 If $X \neq O$, then $1 .' (X \cdot' 1) = 1$.

P6 If P be an atom such that $P \angle X \cdot' , Y$ then there are atoms Q, R such that $Q \angle X$, $R \angle Y$ and $P = Q \cdot' R$.

P7 If P, Q and R are atoms such that $P .' Q \neq O$ $Q .' P \neq O$, $P .' R \neq O$, $R .' P \neq O$, then $Q = R$.

These postulates are satisfied by any realization of the following sort :

Let $\triangle$ be a class of all sets of ordered couples of individuals from some domain of individuals ; let $X \angle Y$ means that X is a subset of Y, let $X .' Y$ be the relative product of X and Y. Such a realization is called a complete atomic relational realization.

DEFINITION 4.1.3.1 An atom P is said to be homogeneous if $P .' P \neq O$

It will be verified that, in the realization described above, a homogeneous atom is a set, whose only member is an ordered couple whose two elements are identical.

The following elementary theorems can trivially be proved.

THEOREM 4.1.3.1 $R \div \Delta \rightarrow R.'0 = 0.'R = 0$

4·13.2 $1.'1 = 1$

4.1.3.3 If P is a homogeneous atom, then $P.'P = P$.

4.134 Δcontains at least one homogenous atom.

4.1.3.5 If P is a homogenious atom and Q is an atom and $P.'Q \pm 0$, then $P.'Q = Q$.

413.6· If P is an atom, then there exists a unique atom Q such that $P.'Q \mp 0$ and $Q.'P \mp 0$. In other words every atom has a unique converse.

4.13.7 If P and Q are distinct homogeneous atoms, then $P.'Q = 0$

4.1.38 Every atom is uniquely expressible in the form $S_1.'1.'S_2$, where S_1 and S_2 are homogeneous atoms.

4·13·9 Let α be the cardinal nnmber of the set $\Delta\alpha$ of all atoms of Δ, and be β be the cardinal number of the set $\Delta\beta$ of all homogeneous atoms. Then $\alpha = \beta^2$

The following fundamental theorem asserts a condition in order that two realizations be isomorphic.

THEOREM 4.1.3.10. Every two realizations of the postulates, which posses the same number (finite or infinite) of homogeneous atoms, are isomorphic.

The next theorems :

Show that all realizations of the above postulates are, of essentially one sort; and establish semi categoricalness of the postulates.

THEOREM 4.1.3.11. Every realization of the above postulates is isomorphic to a complete atomic ralational realization.

THEOREM 4.1.3.12. If the axiom of choice and the generalised continum hypothesis are true, then the above postulates are semi categorical in the following sense :

"A set of postulates is called semi-categorical if every two realizations, which involve the same number of elements are isomorphic.

Mckinsey has also considered the consistency and independence of the postulates by the usual matrix method.

4.2. Matrix development of the algebra of relations: Let $\urcorner$ be a class of elements a, b, c.... By a binary relation on $\urcorner$ is meant any rule R which says, of each ordered pair (a,b) of elements of $\urcorner$ that either the relation R holds between a and b (in symbols a R b) or that it does not hold good in symbol (a $R\overline{b}$). If $\urcorner$ consists of n elements a_1, a_2,—a_n, then we can establish a one-one representation of a relation R on $\urcorner$ by relation matrix $\| r_{ij} \|$ of zeros and ones as constructed below :

$$\begin{array}{c|ccccc}
 & a_1 & a_2 & a_3 & a_j & a_n \\
\hline
a_1 & ra_1a_1 & ra_1a_2 & ra_1a_3 & & ra_1a_n \\
a_2 & ra_2a_2 & ra_na_2 \ldots & \ldots & \ldots & ra_2a_n \\
a_3 & & & & & \\
a_i & & & & & \\
a_n & ra_na_1 & ra_na_2 \ldots & \ldots & \ldots & ra_na_n
\end{array}$$

This is defined by

$$rij = \begin{cases} I \text{ if } i\,R\,j \\ O \text{ if } i\overline{R}j \end{cases}$$

Lifting out the enteriors from the above table, R is associated with the matrix $\| rij \|$.

It is possible to develop an algebra of such matrices.

DEFINITION 4.2.1 $\| rij \| + \| sij \| =_{Df} \| rij + sij \|$

4.2.2 $\| rij \| \cdot \| sij \| \underset{Df}{=} \| rij\ sij \|$

4.2.3 $\| rij \| \cdot' \| sij \| \underset{Df}{=} \| rl\alpha\ s\alpha j \|$, where

$$ri\alpha\ s\alpha j = \sum_{k=1}^{n} rik\ skj.$$

4.2.4 $\| rij \| \angle \| sij \| \underset{D}{=} rij + sij = sij$

Or, rij. sij=rij for every i and j

It is then possible to develop an algebra of relations in the matrix notation as considered above.

4.2.5 $R \angle S \longleftrightarrow \| rij \| \angle \| sij \|$

In other words R is included in S iff when rij=1, then sij=1

4.2.6 $R.S \longleftrightarrow \| rij\ sij \|$.

Thus R.S holds for a pair $\angle$ ij $>$ iff ris sij=1; and this is equivalent to rij=1 as well as sij=1

4.2.7 $R + S \longleftrightarrow \| rij + sij \|$.

Thus R+S holds for $<$i, j$>$ provided that rij+sij=1, i.e. rij=1 or sij=1

4.2.8 $\overline{R} \longleftrightarrow \| r'ij \|$, where $r'ij = 1 - rij$.

Here r'ij=1 or O according as rij=O or 1.

4.2.9 $\breve{R} \longleftrightarrow \| \breve{r}ij \|$, whe $\breve{r}ij = rj\breve{i}ji$

4.2.10 $R.'S \longleftrightarrow \| ri\alpha\ s\alpha j \|$, where

$$ri\alpha\ s\alpha j = \sum_{k=1}^{n} rik\ skj$$

$$= rij\ sij + \ldots\ldots rin\ snj$$

That is R.'S holds for an $<$i,j) $>$ iff $ri\alpha\ s\alpha j = 1$. This sum is equal to 1 iff at least one summand say the kth is equal to 1. Now rik skj=1 iff rik=skj=1. Hence R·'S holds for an $<$ij$>$ iff there is some term k such that $<ik> \in R$ and $<kj> \in S$

4.2.11 $I' \longleftrightarrow \| eij \|$, where

$eii = 1$ and $eij = 0$ if $i \neq j$

4.2.12 $O \longleftrightarrow \| O_{jj} \|$, where
$O_{jj}=O$ for all i and j

4.2.13 $1 \longleftrightarrow \| 1_{ij} \|$, where
$1_{ij}=1$ for all i and j

4.2.14 A relation R is
reflexive iff $1' \angle R$;
symmetric iff $R=\breve{R}$;
|
transitive iff $R.'R \angle R$
diversity iff $R \neq 1'$

The matrix representation of relation on an infinite set can easily be extended. Then under inclusion, the relations on a fixed class ┐ of w_a elements form the atomic Boolean algebra $2^{w_a^2}$.

4.3 Application of the calculus of Relations.

4.3.1 Application in algebra. The calculus of relations includes the elementary theory of groups with relative maltiplication playing the role of group theoretic composition and the converse of a relation corresponds to the group theoretic inverse satisfying $R.'\,\breve{R}=1'=\breve{R}.'R$.

It also includes Boolean algebra. The calculus of relations is thus a union of Boolean algebra and group theory.

4.3.2 Applications in defining numbers. Theory of ratural, numbers can be developed in the calculus of relations [4]

Natural numbers. The natural number zero is regarded as the relation of the zeroth power of a relation A to A itself. Since the zeroth power of a relation A is the relation identity with its field restricted to the field A, it follows that the natural number zero is the relation to A of identity with its field restricted to the field of

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[4] Fredrich Fitch, on natural numbers, integers and ratimals Jour. Sym. Log. 1949.

A. The natural number 1 is regarded as the relation of the first power of a relation A to A itself. In other words, the natural number 1 is simply the relation of identity among relations of some fixed type, since the first power of a relation A is identical with A.

The natural number 2 is regarded as the relation of the square of a relation A to A itself and so on.

The natural numbers are relations of a higher type than the relations they relate.

The numerical product of natural numbers is identified with the relative product of the natural numbers.

We define A to be the R th power as the relations of x to y such that for relation B, it is the case that x bears B to y and that B bears R to A. The numerical sum of two natural numbers R and K can then be defined as the relation A to B such that A is the relative product of B^R with B^K . In other words A is said to be the (R+K) th of B if A is the relative product of Rth power of B with the K th power of B.

4.3.3 Geometric representations of relations [5]. Let us suppose that relation veriables denote relations between real numbers, and let as consider a rectangular co-ordinate system in the plane. Every relation R may then be represented as a certain point set in the plane, namely as the set of all points (x, y) such that x has the relation R to y ; and conversely every point set in the plane represents a certain relation, namely the relation which holds between two numbers x and y if and only if the point (x, y) belongs to the set. The formula 'R=S' is valid in this geometrical representation if and only if the sets corresponding to R and S are identical. The relations 1, O, 1', O' are represented by certain particular point sets : 1 by the whole plane, O by the empty point set, the identity relation by the straight line whose equation is '$x=y$', the diversity relation by the set of all points not on this straight line. The absolute (Boolean) oprations are represented by the usual set theoretic operations on point sets.

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[5] Tarski Alfred, on calculus of relations, Jour sym. log. (1941) 73—89.

In order to obtain the representation for for $\breve{R}$, we take the point set corresponding to R and rotate it in three dimensions through an angle of 180° about the line $x=y$. For the relative operations, we have to resort to threedimensional space and to supplement our co-ordinate system with a z axis perpendicular to the x y plane. Then, in order to obtain the representation of R.'S (in the x y plane), we rotate the point set corresponding to R through a right angle about the x axis, draw through every point of the resulting set a straight line parallel to the y axis, and take the union of all these stright lines; in this way we obtain the "cylindrical" point set $\breve{R}$. Semilarly we rotate the set corresponding to S through a right angle about the y axis, draw lines parallel to the x axis and obtain the "cylindrical" point set $\breve{S}$. Finally we take the intersection of $\breve{R}$ and $\breve{S}$, and project it orthogonally upon the xy plane. The projection thus obtained constitutes the geometrical representation of R.'S. The representation of the relative sum can easily be obtained from that of the relative product in view of the relation between the sum and product in theorem XV.

With the aid of this geometric representation, the content of all axioms for the calculus of relations is easily made intuitively evident.

4.3.4 Solution of a riddle. By the matrix method we shall solve the following riddle. One man looks at a second man and says : Brothers and sisters I have none, but that man's father is my father's son. The question is, what is the relation of the speaker to the second man ? Let us cafine our attention to the four people :

a, the speaker, *b* the second man, *c* speaker's father and *d* the father of the second man. Set up matrices for the relation *S* (son of) and *F* (father of) as follows :

S	a	b	c	d
a	o	x	l	x
b	x	o	x	l
c	o	x	o	o
d	x	o	l	o

F	a	b	c	d
a	o	x	o	x
b	x	o	x	o
c	l	x	o	l
d	x	l	o	o

The ordinary matrix product leads to the matrix associated with SF (son of the father of) :

SF	a	b	c	d
a	l	x	x	l
b	x	l	o	x
c	x	o	x	o
d	l	x	o	l

For SF defined as abovc, as the relation of being a son of the father of , <i, j> ⥶ SF implin i is a brother of j, or i is a sister of j, or i is identical with j. But the speaker a has neither brothers nor sisters. Thus SF table gives us the information that ∠a, a> ⥶SF and ∠ad>⥶ SF. The first is trivial. The second reduce to the assertion that a=d. Hence the speaker is the father of the second man.

Matrices are also used for computing the conclusions traditionally drawn by means of hypothetical syllogisms.

Exercise 4

1 Express the following using the relative producet :

- (i) x is a grand parent of y ;
- (ii) x is a grand son of y ;
- (iii) x is a maternal uncle of y ;

2. What does it mean to say that :

- (a) F.'M (x, y)
- (b) M.'F (x, "y)

(c) S+B (x, y), where F (x, y), M (x, y)
S (x, y) and B (x, y)

stand respectively for—is the father of –, – is the mother of—, —is a selter of –, and—is a brother of —.

3. Prove by matrix method that→=→n for every n in computing the conclusions of hythochethical syllogism.

(Hint—Consider two propositions : $\alpha\rightarrow\beta$ and $\beta\rightarrow\gamma$.

The implication table is ;

$$\begin{array}{c|ccc}
\rightarrow & \alpha & \beta & \gamma \\
\hline
\alpha & | & | & x \\
\beta & x & | & | \\
\gamma & x & x & 1
\end{array}$$

Squaring the resulting matrix, we have :

$$\left|\begin{array}{ccc} | & | & x \\ x & | & | \\ x & x & | \end{array}\right|^{2} = \left\|\begin{array}{ccc} | & | & | \\ x & | & | \\ x & x & | \end{array}\right\|$$

The new table is :

$$\begin{array}{c|ccc}
\rightarrow^{2} & \alpha & \beta & \gamma \\
\hline
\alpha & | & | & | \\
\beta & x & | & | \\
\gamma & x & x & |
\end{array}$$

The reflexiveness and transitivity of the implication relation guarantees that→=→n for every n. The last table shows that we have computed the conclusion $\alpha\rightarrow\gamma$.

4. What are the geometrical representations of R. '1 and 1.'R ?

The relation R.'1 holds cetween x and y if and only if there is a z such that R (x, z). In order to obtain the geometrical representation of R·1, we consider the set corresponding to R, draw through every point of this set a straightline parallel to the y axis, and take the union of all these straightlines. The same method is applied in the case of 1'.R, except that the straight lines are drawn parallel to the x axis.

5. What is the geometrical interpretation of the theorem XII ?

(Hint : It is to verify that either the set representing R.'1 ro the set representing 1.'$\bar{R}$ is the whole plane.

Suppose that the set representing R,'1 is not the whole plane. As we have seen in 4, this set is the union of all straight lines which are parallel to the y axis and which pass through a point belonging to the set which represents R. If this union doesnot coincide with the whole plane, there must be a line parallel to the y axis which doesnot contain any point of the set corresponding to R, and which therefore consists exclusively of points of the set corresponding to $\bar{R}$ If through every point of this line we draw a straight line parallel to the x axis and take the union of all these parallel lines, we obtain the whole plane. Hence the relation 1.'R is represented by the whole plane.)

6. Give the geometrical interpretation of the theorem VII.

(Hint : If we take a point set and rotate it through 180° about a given straight line and then rotate it again through 180° about the same straight line, the result is the original point set.)

7. Give the relation theoretic defination s of integers and rational numbers analogous to the definition of natural members.

(Hint use converse of a relation)

8. Define a triadic and a tetradic relation. Prove that all such relations can be reduced to dyadic (binary) relations.

(Hint : A triadic (ternary) relation may be identified with a ternary propositional function and a tetredic relation with a tretradic propositional function. The relation is said to hold between an ordered triplet of elements or between an ordered quadruplet of elements according as the ternary propositional function is satisfied. But all relations can be defined as binary relations. A triadic relation can be interpreted as a relation of a dyad to an individual and a tretradic as a dyadic relation of dyads, and so on.)

V INTERPRETATION OF MATHEMATICS IN FIRST ORDER THEORIES

1. First order theories.

A theory formalised within FOFC is called a first order theory denoted by FOT. Therefore, a FOT can be factorable into a subtheory which is equivalent to an axiom system of a FOFC, and a remainder with non logical (proper) axioms. Proper axioms vary from theory to theory.

We are interested to formalise some of the mathematical theories. Mathematical theories involue the concept of equality and operations which put them outside the scope of FOFC. But they can be considered as relations. Thus the expression '$x+y=z$' may be considered as a ternary relation $+(x,y,z)$ read as the sum of x and y is z. Equality is considered as a binary relation E provided that

(i) $\pi_{x_1} E(x_1,x_1)$ (reflexity).

(ii) $\pi_{x,y} E(x,y) \longrightarrow E(y,x)$(Symmetry);

(iii) $\pi_{x,y,z} E(x,y) \cdot E(y,z) \longrightarrow E(x,z)$ (transitivity);

(iv) For any nary relation $A(x_1,x_2 \ldots\ldots x_n)$;

$$\pi_{x_1,x_2 \ldots x_n,y_1,y_2 \ldots y_n} E(x_1,y_1) \,.\, E(x_2 y_2) .\, E(x_n y_n) \cdot A(x_1 \; x_2 \ldots\ldots x_n) \rightarrow A(y_1 y_2 \ldots\ldots y_n)$$

(substitutivity)

Any relation E satisfing (1)—(iii) is called an equivalence relation and if it further satisfis (iv), then it is called equality and is denoted by $=(x,y)$ or simply $x=y$.

By a model of a FOT we mean an exterpretation in which all the axions of the system are true. If further E doesnot hold between any two distinct individuals of a model for a FOT, then the model is called normal. Any model M can be contracted to a normal model M^1 by taking the domain D^1 of M^1 to be the set of equivalence classes determined by the relation E in the domain D of M.

Thus the elements a and b belong to the same equivalence class X, if E (a,b) holds in M. Any relation $A(x_1 x_2 \ldots\ldots x_n)$ is defined to hold between individuals $\alpha_1 \alpha_2 \ldots\ldots$ of M^1 i.e. equivalence classes of M if for some, and hence for all elements $a_1 a_2 \ldots\ldots$ an of $\alpha_1 \alpha_2 \ldots\ldots \alpha n$ respectively, $A(a_1, a_2 \ \ldots an)$ holds in M. It is trivial to verify that a wf that holds in M also holds in M^1

Properties of FOT. We state some of the important properties of a FOT without their proofs.

(i) If a closed wf (a formula without free variables) A of a theory T is not provable in T, then the theory T^1, obtained from T by adding A as an axiom, is consistent.

(ii) The set of expressions of a FOT is denumerable, hence the same is true of the set of terms, wfs, closed wfs etc.

FOT is called complete iff for any closed wf A of T either $\vdash_T A$ (A is a theorem in T) or $\dashv_T A$. FOT^1 having the same symbols as a FOT is said to be an extension of T if every theorem of T is a theorem of T^1.

(iii) If T is a consistent FOT, then there is a consistent, complete extension of T. Each consistent decidable FOT has a consistent, decidable complete extension.

(iv) Each consistent FOT has a denumerable model i.e. a model in which the domain is denumerable.

(v) Any FOT which has a model has a denumarable model (Skolem Lo'wen heim Theorem)

(vi) For any two cardinal numbers α and β such that $\alpha < \beta$, if T has a model of cardinality α then T has a model of cardinality β. Hence for any cardinal numbers $\alpha \geq w$ any consistent FOT has a model of cardinality a.

(vii) Any consistent FOT with equality has a finito or denumerable normal model.

(viii) Any FOT with equality which has an infinite normal model M has a denumerable normal model.

If in the notion of FOT, we allow a non denumerable number of functions, individial and functional constans and possibly a non denumerable number of axioms, then we arrive at the notion of a generalised first order theory (GFOT) · FOT^s are special cases of GFOT^s. If the set of symbols of a GFOT can be well ordered and has cardnals $w\propto$, then the set of expressions of T also can be well ordered and has cardinality $w\propto$. If the set of symbols of a consistent GFOT can be well ordeted and has candinality $w\propto$, then T has a model of cardinality $w\propto$.

Sometimes a FOT is called an elementary theory and a FOT with equlity is called an elementary theory with equality. We shall give examples of FOT from Number theory, algebra and geometry.

2. Number Theory. The rigorous foundation of mathematics begins with number theory. The first semi-axiomatic presentation of natural number theory is known as Peon's postulates. Ler (w,s) is a pair consisting of a set w of elements a, b, c...and a function s on w to w called the successor function for which the following three canditions hold :

P1. There is in w an element O such that for each element n of w, $O \neq s(n)$.

P2. $m \in w,\ n \in w,\ s(m)=s(n) \longrightarrow m=n$.

P3. If w^1 is a sub set of w such that $o \in w^1$ and $s(n) \in w^1$ whenever $n \in w^1$, then $w^1 = w$

P3 essentially guarantees the validity of proofs by making mathematical induction as follows.

P3^1. Suppose that for each natural number n S (n) is a statement (for property) true or fale concerning n. If S (n) is true and if S (n+1) is true whenever S(o) ...S(n) are true, then S(n) is true for all natural numbers.

These axions, together with a certain amount of set theory is used to develop the theory of natural rational real and complex numbers.

The axioms involve certain intuitive notions such as 'property' which prevent this system from being a rigorous formalization. More over the principle of the Mathematical induction (P_3) refers intuitively 2^{w} subsets to characterise w elements of natural numbers.

We shall therefore build a $FO{\cdot}T_w$ based on Peon's postulats. T_w has a relation of equatity$=$, one individual constant O and three functions s, $+$and.

The proper axions of the system. $(w, = s, +, \cdot)$.

in FOT are :

1. $=(x_1 x_2) \longrightarrow (=(x_1, x_3) \longrightarrow =(x_2 x_3));$
2. $=(x_1, x_2) \longrightarrow =(s(x_1), s(x_2));$
3. $\overline{=(O, s(x_1))};$
4. $=(s(x_1), s(x_2) \longrightarrow =(x_1, x_2)$
5. $+(O, x_1; x_1))$
6. $=((x_1 + s x_2), s(x_1 + x_2));$
7. $= \times(O, x_1, O)$
8. $=(\times(x_1, s x_2), +(\times(x_1 x_2), x_1));$
9. For any wf A(x), $A(O) \longrightarrow$
   $(\pi_x(\mathbf{A}(x) \rightarrow A\, s(x) \rightarrow \pi_x A_x)$

It is to be noted that the axiom 9 will refer denumarable number of properties defined by wfs of the system. From 9, by modus ponens, it is possible to deduce the induction principle : from A(o) and $\pi_x[A(x) \longrightarrow A\, s(x)]$, we can derive $\pi_x A_x$.

The following stronger forms of the inductions principle can be deduced.

(i) Complete Induction

$$\vdash \cdot \pi_x \cdot \{\pi_y[<(y,x) \longrightarrow A(y) : \longrightarrow A(x)\} \longrightarrow \pi_x A(x)$$

that is, given a property. A such that, for any x, if it holds for all natural numbers len than x, then it holds for x also. Then A holds for all natural numbers

(ii) Least number principle.

$\vdash A(x) \dashrightarrow \Sigma_y \, [A(y) \cdot \pi_z < (zy) \dashrightarrow \overline{A(z) \;]}$

i.e. if a property holds for some natural number, then there is a least number satisfying the property.

The theory developed here is a FOT with equality in which :

$\vdash \cdot = (x_1, x_1)$; $\vdash = (x,y) \dashrightarrow \cdot \, A(x,x) \dashrightarrow A(x,y)$,

where A(x,y) comes from A(x,x) by replacing one or more occurrences of x by y, with the proviso that y is free for those occurrences of x.

It is to be noted that the interpretation in which

(i) the set of non negative integers is the domain

(ii) the intiger o is the interpretation of the symbol **O**;

(iii) the succerror opration (addition of 1) is the interpretation of the function s ;

(iv) ordinary addition and multiplication are the inter pretation of $+$and$\cdot$;

(v) the interpretation of the symbol$=$is the identity relation; is a normal model for the theory and is called the standard model for the theory. Any model for T_w which is not isomsphic to the standerd model is called a non standard model for T_w.

Ehrenfeucht (1) has shown the existence of at least 2^w non-isomosphic models of cardinality ${}^w\alpha$. We have considered consistent new theory T^1 by adding to T_ω a new individual constant ϕ and the axioms $\phi \neq 0$, $\phi \neq 1$, $\phi \neq 2, \ldots b \neq n$ where 1,2...n... are numbers standing for s(0), ss(o)...· (in the appendix).

The existence of non standard models is guaranteed by the following properties of FOT.

(α) Any consistent FOT with equality has a finite or denurmable normal model.

(β) If the set of symbols of a consistent generalised FOT with equality can be well ordered and has cardinality ${}^{w}\alpha$, then it has a normal model of cardinality $<{}^{w}\alpha$.

(γ) If inaddition T has an infinite normal model, then T has a normal model of any cardinality $\omega_{\beta} \geqq {}^{w}\alpha$.

(ι) In particular, if T is an ordinary FOT with equality and T has an infinite normal model, then T has a normal model of any cardinality ω_{β}.

Presburger [2] has considered the completeness and decision problem of a new system T'_{ω} obtained from T_{ω} by elliminating the function· and the axions 7 and 8 of T_{ω}. (Vide the appendix I)

2. Elementary theory of group, Ring and field.

2.1 Group· A system (G,E,+) where G is a set of elements E is a binary and+is a ternary relation read as (x equals y, z is the sum of x and y) is said to be a group provided the following proper axioms are satisfied.

(i) axioms of equality.

(ii) $\pi_{x,y}\, \Sigma_{z}[+(x,y,z)]$(existence of the sum)

(iii) $\pi_{x,y,z,\omega}[\ +(x,y,z)+(x,y,w) \longrightarrow z=w\ ]$ (uniqueness of the sum)

(iv) $\pi_{x,y,z,t,u,v}[\ +(x,y,z)+(z,t,u)+(x,y,t,v)\rightarrow +(x,v,u)\ ]$ (Accociativity)

(v) $\Sigma_{e}\pi_{y}\Sigma_{y^{1}}[+(e,y,y)+(y^{1}ye)]$(Existance of a left neutral element w.r.t. which each element possesses a left inverse

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[1] Ehrenfeucht, A. Theories having at least continum many non-somorphic models in each infinite power notices Am. Math. Sol. Vol 5 (680)

[2] Presburger ueber dic vollstan dog keit lines gewissen system, cemptes Rendes, T congres des Meth des pays Slaves Warsaw 192—201, 39

Or

(v^1) Let e be an individual constant in G such that

$$\pi_y \Sigma y^1 [+(e,y,y) \cdot +(y^1,y,y^1,y,e)]$$

Thus we have two elementery theories ($1-v$) and ($1-v^1$) of a group i.e. first order theory with equality. If one adds to the axioms the following wf :

(vi) $\pi_{x,y,z}[+(x,y,z) \rightarrow +(y,x,z)]$, the new theory T_{ga} is called the elementary theory of abelian groups.

2.2 Ring. Let R be a system with three relations $=(x,y)$, $+(x,y,z)$ and $\times(x,y,z)$. Then the system is a ring provided that :

1. R is an ablian group with $+$;
2. The II, III and IV axioms of 2·1 with $+$ replaced by $\times$ hold good.
3. $\pi_{x,y,z,t,u,v,w}[+(x,y,z) \cdot +(z,t,u) \cdot \times(x,t,v) \times(y,t,w) \rightarrow +(v,w,u)]$

   $\pi_{x,y,z,t,u,v,w}[+(x,y,z) \cdot \times(t,z,u) \times(t,x,v) . \times(t,y,w) \rightarrow +(v,w,u)]$ (Distributive law)

If we add the additional

axiom :

4 $\pi_{x,y,z}[\times(x,y,z) \rightarrow \times(y,x,z)]$, the ring is a commutative ring

1.23 field. A set of axioms for a field is obtained by adding the axioms of a ring with the following axioms.

(i) $\Sigma_{x,y}[\overline{=(x,y)}]$

(ii) $\pi_{x,y}\ \Sigma_z[+(x,x,x) \vee \times(x,z,y)$

(iii) $\pi_{x,y} \Sigma_z[+(x,x,x) \vee \times(z,x,y)]$

The notion of a commutative field can also be introduced.

Characteristic of a field. For this purpose we define predicats S_n $(x,y), n=0,1,2,\ldots\ldots$ recursively.

$S_o(x,y) =_{D} +(y,y,y)$;

$S_n(x,y) = \Sigma_z[S_{n-1}(x,z) \cdot +(z,x,y)]$ $n=1,2\ldots$

It is to be remarked that $S_n(x,y)$ is equivalent to $y=nx$ (the result of adding x to it self n times) and So (x,y) characterises the neutral element o)

Let C_n be a wf :

$Cn = \pi_{x,y}[Sn(x,y) \rightarrow +(y,y,y)]$ $n=1, 2,3\ldots$

i.e. Cn states that $nx=0$ for all n.

If p is a prime, the notion of a commugatives field Fp of characteristic p is obtained by adding C_p to the theory of commutative field F.

F_o (a field of characteristic zero) is obtained from Fp by adding as axioms $C_2, C_3, \ldots\ldots C_p \ldots\ldots$ for all primes p.

But the concept of a commutative field of characteristic o cannot be formalised by a finite number of axioms within FOFC.

For the notion of an ordered field, ord F, we introduce an additional two place relasion $\leqslant(x,y)$ read as x is equal to or smaller than y and we add the following axioms to theory of F.

$\pi_{x,y,z,s}[\leqslant(x,y) \longrightarrow \leqslant(+(x,z),+(y,z)]\leqslant(+(z,x),+(z,y))]$

$\pi_{x,y,z}[\leqslant(x,y)\leqslant(o,z)\rightarrow\leqslant(\times(x,z),\times(y,z))\cdot.\leqslant(\times(z,x),\times(z,y))]\cdot.$

where o is the additive neutral element.

For an Archimeden ordered field we introduce a predicate $Q_n(x,y)$.

$Q_n(x,y) = \Sigma_z[Sn(x,z)\cdot\leqslant(y,z)]$, i.e. $nx \leqslant y$ for any natural number n :

An Archimedean ordered field is obtained from an Or F by adding the axiom :

$$\pi_{x,y}[=(o,x) V Q(\overline{o,x}) \cdot V Q_2(x,y) V \cdots\cdots]$$

3 Elementary geometries.

An elementary geometry is that part of geometry which can be formalised and established without the help of any set theoretical device. It can be formalised within the first order theory.

3·1 Euclidian Geometry. We shall consider two systems for elementary Euclidian plane geometry. Besides the logical symbols, the systems contain identty relation $x=y$ diversity relation $x \neq y$, a ternary relation β and a quaternery relation δ. Intuitively $\beta(x,y,z)$ has the meaning x y and z are collinear and y is between x and z, while $\delta(x,y;z,w)$ has the meaning that segment xy is congruent to the segment zw.

Let $(G,=,\neq,\beta,\delta)$ be a system with the following axioms :

(i) $\pi_{x,y}\ \beta(xyx) \rightarrow x=y$

(identity axiom for betweenness)

(ii) $\pi_{x,y,z,u}[\beta(xyu) \cdot \beta(yzu) \rightarrow \beta(xyz)]$

(transitivity axiom for betweenness.)

(iii) $\pi_{x,y,z,u}[\beta(x,y,z) \cdot \beta(xyu) \cdot x \neq y \rightarrow \beta(xzu) \vee \beta(xuz)\ ]$

(connectivity axiom for betweenness)

(iv) $\pi_{x,y}\ \delta(x,y;\ y,x)$ (reflexivity of the congruence of segments)

(v) $\pi_{x,y,z}[\delta(x,y;zz) \rightarrow (x=y)]$

(vi) $\pi_{x,y,z,u,v,w}[\delta(xy;\ zu) \cdot \delta(xy;vw) \rightarrow \delta(zu\ ;\ vw)$

(Transitivity of the congruence relation between segments)

Interms of the relations β and δ it is possible to introduce the notion of collinearity $\lambda(xyz)$ and parallelism $\mu\ (xy\ ;\ uv)$.

DEFINITION 1 $\lambda(xyz) = \beta(xyz) \vee \beta(yzx) \vee \beta(zxy)$

$$2\quad \mu(xy\,;\,uv) \underset{D}{=} \overline{[\Sigma t\,\lambda(xyt)\cdot\lambda(uvt)]},$$

i.e., $\mu(xy;uv)$ states that (x,y) and (u,v) are pairs of distinct points lying on distinct parallel lines.

(vii) $\pi_{txyzu}\Sigma_V[\beta(ztu)\rightarrow\lambda(ytv).[\beta(zvx)\vee\beta(uvx)]$

(strong form of Pasch's axiom)

(viii) $\pi_{t,x,y\cdot z,u,}\Sigma_{V,w}[\beta(x,u,t)\cdot\beta(yuz)$
$x\neq u\rightarrow\beta(xzv)\cdot\beta(xyw)\beta(vtw)]$ (Euclid's axioms)

(ix) $\pi_{x,y,z,u,x^1,y^1,z^1,u^1}[\delta(x,y\,;\,x^1y^1)$
$\delta(y,z;y^1z^1)\delta(xu;x^1u^1)\cdot\delta(yu\,;\,y^1u^1)$
$\beta(xyz)\beta(x^1y^1z^1)\cdot x\neq y\rightarrow\delta(zu\,;\,z^1u^1)]$

(congruence of five segment axiom)

(x) $\pi_{x,y,u,v}\,\Sigma_z[\beta(xyz)\cdot\delta(yz\,;\,u,v)]$

(segment construction axiom)

(xi) $\Sigma_{x,y,z}[\overline{\lambda(x,y,z)}]$

THEOREM 1. viii$\longleftrightarrow$viii^1 where
viii^1 : $\pi_{x,y\cdot z,uv}[\mu(x,y\,;\,z,u)\;\mu(xy\,;\,zv)\rightarrow\lambda(z,uv)$

A system $(G=,\neq,\beta)$ with axiams i→xi is called an elementary Euclidian plane gedmetry G. The existence axioms in G gurantees the existence of those points which are the intersections of lines and those that can be constructed by the use of a transfer segment (x).

An analogue of Archimedean and completeness axiom is also added

(xii) $\pi_{x,y,z},x^1,y^1,u\Sigma_{y^1}[\delta(ux,ux^1)\delta(uz\,;\,uz^1)\cdot$
$\beta(uxy)\cdot\beta(xyz)\rightarrow\delta(uy,uy^1)\beta(x^1y^1z^1)\,]$

A geometary with axioms i—xii is denoted by G^1

We shall next consider models of G and G^1. If F is an ordered field, we define the two dimensinal co-ordinate geometry $\Sigma(F)$ as the set of all ordered pairs $x=(x_1,x_2)$ of F with

$\beta(xyz) = _{D} [(x_1 - y_1)(y_2 - z_2) = (x_2 - y_2)(y_1 - z_1)$ and

$0 \leqslant (x_1 - y_1)(y_1 - z_1)$ and $0 \leqslant (x_2 - y_2)(y_1 - z_2)]$;

$\delta(xy\,;zu) = \ [(x_1 - y_1{}^2) + (x_2 - y_2)^2 = (z_1 - u_1)^2 + (z_2 - u_2)^2]$

If F is a Pythagorean field i. e. the sum of two squares is a square), then Σ (F) is a model for G· Conversely, any model for G is isomorphic to Σ (F) for some pythagorean field F.

The models for G′ are isomorphic to the geometries Σ(F) when F is Euclidian, i.e. has the property that every positive element is a square. Conversely each such geometry is a model for G^{I}

3'2 Ellipic Geometry, A similar set of axioms for elliptic plane geometry can be given except that order is now expresed by means of a four place relation γ (xy ; zw) with the meaning that x,y,z and w arc collinear and the pair (x,y) doesnot separate the pair (z,w) There will be two systems E and E^1 depending on whether or not we include the axiom corresponding to E(xii) as follows :

E(xii) $\pi_{x,y,z,w}\gamma(xyzw) \longleftrightarrow$

$\Sigma_{r,s,t,u,v}[\lambda(xyz)\lambda(yzw) \cdot \lambda(xyt) \cdot \lambda(xuv)$

$\lambda(wrs) \cdot \lambda(uyr) \cdot \lambda(uts)\ \lambda(vtr)\lambda(vzs)]\}$

Let F be a Pythogarean field. Then by the elliptic geomatry Σ(F) we mean the set of ordered triples $x = (x_1, x_2, x_3) \neq (0,0,0)$ of F, where $(ax_1,\ ax_2,\ ax_s)$ is taken to be equivalent to (x_1, x_2, x_3) for $a \neq 0$

DEFINITION $\lambda(x,y,z)$ is defined to mean that the triple x,y and z are linearly dependent,

δ(xy,zw) is meant that

$$\frac{(\Sigma_1{}^3 xiyi)^2}{\Sigma_1{}^3 x_1{}^2 \Sigma_1{}^3 y_1{}^2} = \frac{(\Sigma_1{}^3 z_1 w_1)^2}{\Sigma zi_i{}^2 \Sigma w_i{}^2}$$

The notion γ can then be defined in terms of the order in F so that E(F) becomes a model for E and all model cf E are (isomorphic) to E(F) for some F. The geometry E(F) is a model for E^1 iff F is Euclidian.

DEFINITION 3·2·1 In elliplic geometry it is possible to introduce the binary relation $\alpha(x,y)$ of polarity between points which indicates that one point lies on the polar of the other. $\lambda(x,yz) \longleftrightarrow \sum_t [\alpha(t,x) \cdot \alpha(t,y)\alpha(t,z)]$

i.e. collinearity is introduced in terms of α.

3·3 Hyperbolic geometry. By the elementry hyperbolic geometry H we mean the geometry which follows from axioms I—VII and IX—XI together with the negalition of VIII. If XII is also assumed then we call the geometry H^1.

The notion of μ which were defined for G and G^1 would mean non intersection rather than parallelism. Parallelism μ^1 is defined as follows.

$$\mu^1(xy\,;\,zw) = \quad \pi_u \Sigma_V \{\mu(\mu y; zw).\ \beta(xuw) \rightarrow \beta(zuv) \cdot \lambda(xyv)\}$$

In H the axiom XII←→HXII which asserts the existence of parallels :

$$\mathrm{HXII}\ \ \pi_{xyz} \Sigma_w [\overline{\lambda(xyz)} \rightarrow \mu^1(xy,zw)]$$

Let F be a pythogorean field and e be a positive element in F such that for each x,y in F with $x^2+y^2<e$, there is a $z \in F$ such that $z^2=e-x^2-y^2$. Then a model F(f,e) forH is obtained by taking all pairs $z=(x_1,x_2)$ of elements from F subject to the restriction $x_1{}^2+x_2{}^2<e$, where the basic relations are defined by the conditions :

$$\beta(xyz) \underset{D}{=} \quad [(x_1-y_1)(y_2-z_2)=(x_2-y_2)(y_1-z_1)$$

$$\text{and } O \lesseqgtr (x_1-y_1)(y_1-z_1) \text{ and } O \leqslant (x_2-y_2)(y_2-z_2)]$$

$$\delta(xy\ ;\quad z,u) = \left[\frac{(e-x_1y_1-x_2y_2)^2}{(e-x_1{}^2-x_2{}^2)(e-y_1{}^2-y_2{}^2)} = \frac{(e-z_1u_1-z_2u_2)^2}{(e-z_1{}^2-z_2{}^2)(e-u_1{}^2--u_2{}^2)}\right]$$

Every model of H is isomosphic to some H(F,e). If e is a square, then F is Euclidian and by a change of coordinates we may take e=1. Every model of H is isomosphic·

H(F)=H(F,I) for some Euclidian field F·

Exercise 8

1 Show that the induction principle I in T_w is independent of the other axioms

(Hint : Consider the enterpretation having as its domain the set of polynomials with intergral coefficients such that the leading coefficient is non negative; the axioms 1−8 hold with the usual operation of addition and multiplication, but the wf :

$$\vdash \cdot\ y \neq o \rightarrow \Sigma_{u1v}\ [x = yu + v \cdot v < y]$$

is false by sub stituting the polynomial x for x and 2 for y·

2. By the ordinary mathematical reasons solve the finite marriage problem : Given a finite set M of m men and a set W of women such that each man knows only a finits number of women and, for $1 \leqslant k \leqslant m$ any subset of M having k elements is acquainted with at least k women of W i.e. there are at least k women acquainted with at least one of the k given men. Then it is possible to marry monogamous by all the men of M to women in W so that every man is married to a woman withwhom he is acquainted.

3. Give a geometrical model of a lady's ornament S consisting of a set of beads arranged on wires in accordance with the instructions, called axioms ;

   (1) Each pair of wires has at least one bead in common;

   (2) Each pair of wires has not more than one bead in common;

   (3) Each bead in S is on at least two wires;

   (4) The total uumber of wires in S is four.

(**Hint** : The ornament system may be linked with two axiomatic finite geometries by replacing the words : ornament, bead and wire by the geometrical elements :

   (i) Plane, point and line;

   (ii) Plane, line and point.)

4. Give a geometrical model of the Bhagalpur University U consisting of a set of committees formed for all sorts of works in accordance with the following instructions of the V. C., called axioms :

   (i) If M_1 and M_2 are different members of U, then there exists at least one committee containing both M_1 and M_2;

   (ii) If M_1 and M_2 are different members of U, there is not more than one committee containing of both M_1 and M_2;

   (iii) Any two committes have at least one member in common;

   (iv) There exists at least one committee;

   (v) Each committee contains at least three members of U.

   (vi) Not all members serve on the same committee.

(**Hint** : Replace the words Unisversity, committees and members respectively by plane, lines, points.)

3. Give a formalization of an elementry r-dimentional elliptic and hyberbolic glometry.

APPENDIX

AXIOMATIZATION AND ARITHMETIC OF THE NATURAL NUMBERS

Here we propose to examine

(i) the axiom systems for the natural numbers,

(ii) the construction of their arithmetic and

(iii) the consistency as well as the decision problem in it.

(i) *Axoim Systems For the Natural Numbers.*

We shall examinc here the axiom systems for the natural numbers of Peano, Germansky,[1] Schmidt[2] and Wang[3].

(*a*) *System of Peano*

Let us assume the *existence* of a system (N, s), where N is a nonempty set of elements m, n, p.........and s is an unary operation, possessing the properties called axioms to be listed below :—

1. *s* is a single valued function on N to N or a mapping of N into N, that is, to each element *n* in N corresponds a uniqüly detemined element *s n*, called the success or of *n*.

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(1) Baruch Germansky, (i) "Axiom of Natural Numbers" Lematimat'ka Vol. 1 June (1947).
(ii) Supplement to the paper "Axioms for the Natural Numbers", Ibid, Vol. 3 (1249), p. 8.
(iii) "A new set of axioms sufficient for the development of the theory of natural numbers" Ibid pp. 65:67

(2) Hans Rohr Baruch "Das Axiomen system Von E hard Schmide fur menge der naturlichen Zahlen.";
Methematische Nachrichten Vol. 4 (1951) pp.315-21.

(3) Shianghaw Wang, "A system of completely independent axiom for the sequence of natural numbers", Jour. Sym. Logic (1943) p. 84.

2. $\exists n_0 \, [n. \longrightarrow . \, sn \neq n_0]$,

that is, there is an element n_0 in N which is not successor sn in N.

3. $(m,n) \, [sm = sn. \longrightarrow . m = n]$,

that is, there are no two elements with the same successor.

4. $(M) \, [M \subset N. \, sM \subset N.M. \, [N - sN] \neq \phi : \rightarrow M = N]$

that is, if M is a subset of N and contains the successor of each of its elements and furthermore contains an element n_0, which is not the successor of an element in M, then M is the whole of N. This axiom my be characterized as the "Principle of Induction for Natural Numbers"

(b) Systems of Germansky

Germansky has proposed a series of four axiom systems for the development of the theory of natural numbers which will be denoted here by G (1), G (2), G (3) and G (4).

G(1). Let us assume the existence of a system (N, R), where N is a non-empty set of elements $m, n, p, \ldots\ldots$ having a relationship R (called adjacency or neighbouring relation), possessing the properties called axioms—to be listed below :—

1. $n_0 \in N$, that is, the set N is not empty, it contains an element n_0.

2. $(n_1, n_2) \, [n_0 R n_1 . \, n_0 R n_2 : \longrightarrow . n_1 = n_2]$,

that is, for every n_1 and n_2 belonging to N, $n_0 R n_1$ and $n_0 R n_2$ imply $n_1 = n_2$. In other words, there exists exactly one element n_0 in N possessing only one adjacent element distinct from itself.

4. $(m,n) \, [mRn. \rightarrow . m \neq n]$,

that is, an element cannot be adjacent to itself.

5. $(m,n) \, [mRn. \rightarrow nRm]$,

that is, the relation of adjacency is symmetric.

6. $(A,B) \, [AB = \phi . \, A + B = N : \rightarrow (\exists m,n) \, m \in A. \, n \in B. \, mRn)]$

that is, every subdivison of the set N in to two non-empty parts has the property that there exists an element, in each part adjacent to an element in other part.

G(2). The axioms of G(2) are the same as of G (1) except that the last axiom of G(1), i. e, 6 is replaced by the following axiom $6x$:

$6x.$ $(m, \mathrm{M})\ [\mathrm{M} \sqsubset \mathrm{N}.\ m \in \mathrm{M}.\ m\mathrm{R}n \rightarrow n \in \mathrm{M}.\ \mathrm{M} = \mathrm{N}]$,

that is, if a non-empty subset M of N is closed w. r. t. the relation of adjacency, it follows that $\mathrm{M} = \mathrm{N}$.

The two axiom systems G(1) and G(2) are the same. in G(1) the 'Induction Principle' is introduced as a consequence of the axioms, while in G(2) it is introduced by a separate postulate $6x$.

In G(1) and G(2) we have not taken into consideration left or right adjaceney. But If we take this into consideration, adjacency can be split up in to left and right adjacency. Germansky has developed systems G(3) and G (4) with these primitive ideas.

G(3). Let us assume the existence of a system (N, R′), where N is a set of elements m, n — — having a relation R′ (called left adjacency) and possesing the following properties :—

1. The set N is infinite.

2. There is a definite element n_0 of N having the property, that whenever $m \in \mathrm{N}$ and $m \neq n_0$, there exists a unique n such that $n \in \mathrm{N}$ and $n\mathrm{R}'m$ and for this n, we have also $n \neq m$.

5. If a non-empty subset M of N has the property that whenever $m \in \mathrm{M}$ and $n\mathrm{R}'m$, it follows that there exists one and only one element m_0 of M for which there is no t such that $t \in \mathrm{M}$ and $t\mathrm{R}'m_0$.

G(4). Parallel to the axiom system G(3), he introduced also a similar set of axioms G(4) in which N is assumed only to be non-empty and the relation of left adjacency is replaced by the relation of right adjacency.

(c) System of Schmidt

Let (N, R) be a system, where N is a non-empty set of elements and R is the ordering relation in N, possessing the following properties · :

1. Any non-empty subset of N has a first element that is, N is a well-ordered set.

2. Any element of N, with the exception of the first has an immediate predecessor, which also belongs to N.

3. N has no last element, that is, N is simply infinite.

It is to be observed that the concept of 'first', 'last' and 'predecessor' which appear in the axioms is assumed to be contained in the concept of order.

(d) Wang has also proposed a system of completely independent axioms for the sequence of natural numbers. But this axiom system has a strong resemblance to the systems G(3) and G(4) of Germansky.

System G(3) of Germansky has also a strong resemblance to the axiom system of Schmidt.

Now all these axiom systems are equivalent to the Peano System because they are mutually derivable.

Now the problem before us is to examine the categoricalness of the axiom systems. As all the axiom systems are equivalent to Peano System, it is only sufficient if we examine the categoricalness of Peano axioms.

Let n_0 stand for 0 (having the usual meaning) and sn_0 for 1, then a model M for the Peano axioms will contain a sequence of order type 0, 1, 2,.........n,........and using the axioms 1, 2 and 3, it can easily be seen that all the elements are distinct and without other predecessors. We call this as the "Sequence of Natural Numbers".

Axiom 4 (Principle of Induction) is usually considered te determine the categoricalness of the system of axioms. The usual proof runs somewhat as follows :--

"M has been shown to contain a sequence of natural numbers. Can M contain elements other than the sequence of natural numbers ? If possible, let us suppose that M contains elements other than the sequence of natural numbers. In M, consider a set

N, consisting of just the natural numbers. Since N contains zero, and is closed under the successor function, we infer from the axiom of complete Induction, that N contains all the elements of M, contrary to hypothesis that some elements of M were not natural numbers".

The following objections can be raised as regards the above alleged proof of the categoricalness of the Peano axioms.

(1) The above proof of categoricalness has its basis on the notion of the standard meaning of a set, as it appears in the axiom of Induction. But in an axiomatic system, the meaning of a set has not always its standard meaning. In an axiomatic system where natural numbers 0, 1, 2,........n........are constructible, *a set like* what is considered to be a valid set in intuitive system, may not be constructible. And hence it will be meaningless to speak of as a subset of the set of natural numberes.

(2) The second objection is with regard to the nature of models. Besides standard models, nonstandard models can also be constructed which contain elements not necessarily isomorphic to the sequence of natural numbers. From the following theorem of Leon Henkein,[4] nonstandard models are constructed.

Theorem 1. A set T *of* cwffs (closed well-formed formulas) is satisfiable w.r.t. some model of denumerable elements if and only if every finite subset of T is satisfiable.

Now Peano system does not limit the concept of primitive constants, hence either a new primitive constant or a non-denumerable number of primitive constants may be adjoined.

Let us first, adjoin a new primitive constant ϕ and add to the given set of axioms, the infinite list of formulas (axioms) $\phi \neq 0, \phi \neq 1, \ldots \ldots \phi \neq n \ldots \ldots$(where ϕ may be anything except natural numbers). Any finite subset of the enlarged system of axioms is clearly satisfiable. Hence, from Th. 1, it follows that a model of

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(4) Leon Henkein, "Completeness in the Theory of Types" J. Sym. Logic (1950) pp. 81-91.

dennmerable elements satisfies the full set of formulas. We can even adjoin a non-denumerable number of primitive constants ψ^{α} together with all formulas $\psi_1^{\alpha_1} \neq \psi_2^{\alpha_2}$ for $\psi_1 \neq \psi_2$ and thus construct models for which Peano axioms are valid.

From the above, we conclude that axiom systems for the natural numbers are noncategorical and generally no mathematical axiom systems can be genuinely categorical, unless its models have some definite finite cardinal number, and provided that the logical notions of set and functions are axiomatized along with some specific methematical notions.

It is also interesting to note the following two remarks of Skolen[5] as regards the characterization of natural numbers.

(i) It is impossible to characterize the natural number sequence by finitely many axioms, formulated within the first order functional calculus.

(ii) A new interpretation will always be possible no matter how many properties are used to set up the concept of natural numbers.

Thus we conclude that the sequence of natural numbers assumes an aspect of obscurity and deficiency when viewed from an axiomatic angle.

(*ii*) *Arithmetic of Natural Nambers*

We next propose to examine the construction of the arithmetic of natural numbers in one of the axiom systems, namely in Peano system.

In the usual construction of arithmetic, addition as well as multiplication is defined recursively. But Motokinti[6] Kendo has

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(5) Skolen (i) "On the impossibility of completely characterizing the number series by means of a finite system of axioms. (title translated from the Norwegian), Norsk. Math. Forenings Skrifter, Ser II. 1933.

(ii) "On some questions regarding the foundation of Mathematics" (title translated frome the Norwegian), Skrifter Norsk. Vid. Akd. Osio, 1 Meth. Nat. K I 1929.

(6) Motokinti Kendo, Use Methode operationalle dans la theorie "des nomberes naturals", Proc. Imp. Acd. Tokyo (Oct. 1944), pp. 564-68.

developed the arithmetic of natural number in which altogether a different course has been followed. Instead of defining addition and multiplication recursively, certain operations are determined inductively, in terms of which addition and multiplication have been defined, Here addition is defined as a certain linear operation and multiplication as a certain additive operation.

Throughout we shall assume that the variables $x, y, z, \ldots\ldots a, b, c$, ...range over the class of natural numbers.

Dfn. 1. An operation $L(x)$ is linear if for every natural number x, $L(x')=(L(x))'$, where x' is sx, the succecsor of x. For example $f(x)$ and $I(x)$ are liner operations if $f(x)=x$ and $I(x)=x$.

Theorem 2. $L(x)$ is determined completely by $L(1)$, i. e., the value of the liner operation at 1.

Proof :— Let $L_1(x)$ and $L_2(x)$ be any two linear operations defined over N and let M be the class of all naturals x such that $L_1(x)=L_2(x)$.

Since $L_1(1)=L_2(1)$, it follows that $1 \in M$. Again let us suppose that $x \in M$, i. e., $L_1(x)=L_2(x)$. Therefore, $(L_1(x))'=L_2(x))'$ and consequently $L_1(x')=L_2(x')$ and hence $x' \in M$ also. Now since M contain 1 and is closed under the successor function, it follows from the axiom of complete Induction that M is the class of all natural numbers. Hence it follows that if the value of linear operation are equal at the natural number 1 they are identically the same.

The value of the linear operation at the natural numder 1 will determine the nature of the operation. When $L(1)=1$, the corresponding operation is given by $L(x)=x$. When $L(1)=1'$, the corresponding operation will by $\phi_1(x)$ where $\phi_1(x)=x'$ and generally the linear operation will be denoted by $\phi_a(x)$ when $L(1)=a'$.

Dfn. 2. c sum $(a, b) \cdot\equiv\cdot (a, b)\ \Sigma_c[\phi_c(x)=\phi_a\phi_b(x)]$, that is, for every natural number a and b, there exists a natural number c, such that $\phi_c(x)=\phi_a\phi_b(x)$ defines the sum of two natural numbers.

Since ϕ_a and ϕ_b are linear it follows that $\phi_a\phi_b$ is also linear; c sum (a, b) will be also be denoted by $c=a+b$.

Properties of $+$

1. $+$ is single valued, i.e., $(a, b, c, d)\ [a=b \cdot c=d :\rightarrow . a+c=b+d]$

Proof :— $a=b \rightarrow \phi_a(x)=\phi_b(x)$ and
$c=d \rightarrow \phi_c(x)=\phi_d(x)$.

Therefor, $\phi_a\phi_c(x)=\phi_b\phi_d(x)$ and consequenty $a+c=b+d$.

2. $+$ is associative, that is,

$(a, b, c)\ [(a+b)+c=a+(b+c)]$.

For liner operations are associative, that is,
$(\phi_a\phi_b)\ \phi_c(x) \quad = \quad \phi_a(\phi_b\phi_c(x)$.

3. $+$ is commutative, that is,
$(a, b)\ [a+b=b+a]$.

The proof of the commutative law is based on the following lemmas.

Lemma 1. $1+a=a+1 \quad = a'$.

For $\phi_{1+a}(x)=\phi_1\ \phi_a(x) \quad =(\phi_a(x))'$
$=\phi_a(\phi_1\ (x)) \quad =\phi_{a+1}(x)$.

Further $\phi_{1+a}\ (1)=(\phi_a(1))' \quad =(a')'$.

Proving the result $1+a=a+1=a'$.

Lemma 2. $a'+b=a+b'=(a+b)'$,

From 2, and Lemma 1, we have

$a'+b=(a+1)+b=a+(1+b)=a+b'=a+(b+1)$
$=(a+b)+1=(a+b)'$.

Now we shall prove the commutative property of $+$. Let us consider two operations $f(x)$ and $g\ (x)$, where $f(x)=x+b$ and $g(x)=b+x$. Each of these operations are linear. For $f(x')=x'+b=(x+b)'=(f\ (x)'$ and similarly $g(x')=(g(x))'$.

Further, by Lemma 1, $f(1)=1+b=b+1=g(1)$ and consequently $f(x)=g(x)$ and also in particular $f(a)=g(a)$. Hence $a+b=b+a$, for a given b and all a. Similarly a second application, of the Induction principle will finally prove that $(a, b)\ [a+b=b+a]$.

Next an additive operation is defined.

Dfn. 3. An operation is additive, denoted by $A(x)$, if for every natural numbers a and b, $A\ (a+b)=A(a)+A(b)$. For example $P(x)$ and $1(x)$ are additive if $P(x)=x+x$ and $1(x)=x$.

Theorem 3. $A(x)$ is determined completely by $A(1)$. The proof of the theorem follows from arguments similar to those of theorem 2.

When $A(1)=a$, $A(x)$ will be denoted by $\psi_a(x)$.

*D***fn. 4.** c Prod. $(ab)\ .\equiv.\ (a, b)\ \exists_c\ [\psi_c(x)=\psi_a\psi_b(x)]$, that is, for every natural numbers a and b there exists a natural number c, such that $\psi_c(x)=\psi_a\psi_b(x)$ defines the product of two natural numbers.

As ψ_a and ψ_b are additives, it follows that $\psi_a\psi_b$ is also additive.

c Prod (ab) will be also denoted by $c=a.b$.

Sometimes the sign of multiplicative dot· will be dropped in order not to be confused with the corresponding logical dot for conjunction.

Properties of·.

1. · is single valued that is·,

$(a,\ b,\ c,\ d)\ [(a=b.c=d:\rightarrow.ac=bd]$.

proof of which follow from arguments similar to those as given in the corresponding property of $+$

2. . is associative, that is,

$(a,b,c)\ [(a(bc)=(ab)c]$

For the operations are associative, i.e.,

$$\psi_a(\psi_b\psi_c)=(\psi_a\psi_b)\psi_c$$

3. . is right and left distributive over $+$, that is, $(a,b,c)\ [a(b+c)=ab+ac.\ (b+c)\ a=ba+ca]$.

For $\psi_a(b+c)=\psi_a\psi_{b+c}=\psi_a(\psi_b+\psi_c)=\psi_a\psi_b+\psi_a\psi_c$ and consequently $a(b+c)=ab+ac$.

Similarly it can be proved that $(b+c)\,a=ba+ca$.

4. $\cdot$ is commutative, the proof of which is based on the following Lemmas.

Lemma 3. $(a)\,[a1=1a=a]$.

Lemma 4. $(a, b)\,[a'b=ab+b.\ ab'=ab+a]$.

Proof af the commutativa property :

Let $F(x)$ and $G(x)$ be the two operations, where $F(x)=xb$ and $G(x)=bx$ and it can be easily proved that both are additives. By Lemma 3, $F(1)=1b=b1=G(1)$ and consequently $F(x)=G(x)$. In particular $F(a)=G(a)$ and consequently $ab=ba$ for a given b and all a. Similarly a second application of the Induction principle will finaly prove that $(a,b)\,[ab=ba]$.

We have based the arithmetic of nutural numbers on certain linear and additive operations in terms of which addition and multiplication are defined. In this construction of arithmatic, there is a technical advantage of eliminating recursion at each step in the definition of $+$ and $\cdot$. Once we define and determine linear and additive operations inductively, we need not repeat recursion further in defining $+$ and $\cdot$.

But arithmetic of Motokindi Kendo cannot be said to be a new thing. He has only indentified the linear and additive operations with the following two trans formations of the additive and multiplicative system of natural numbers.
Here,

$$\phi_a(x) \longleftrightarrow T_a \begin{pmatrix} 0, 1, \ldots 2, \ldots 3, \ldots x \ldots \ldots \ldots \\ a, a+1, a+2, a+3 \ldots a+x \ldots \end{pmatrix}$$

$$\psi_a(x) \longleftrightarrow T_a{}^1 \begin{pmatrix} 0, 1, 2, 3, \ldots\ldots x \ldots\ldots \\ 0, a, 2a, 3a, \ldots\ldots xa \ldots\ldots \end{pmatrix}$$

Thus $\phi_a\,\phi_b \longleftrightarrow T_a \cdot T_b \quad = a+b$

$\psi_a\,\psi_b \longleftrightarrow Ta^1 \,.\, Tb^1 = ab.$

(iii) Consistency and Decision Problem In The Arithmetic of Natural Numbers.

We next propose to examine the consistency and decision problems in the arithmetic of naturul numbers.

Consistency :—A system can be called consistent if every formula of that system is not provable in that system. This amounts to saying that no two formulas, one of which is the negation of the other, are both provable in that system

For a certain elementary system of axioms concerning natural numbers, consistency was established successfully by Hilbert[7] and Bernays. But his bright hopes for the consistency of the entire arithmetic of natural numbers were nipped in the bud due to "Gödel's Incompleteness[8] Theorems". Gödel showed that every consistent formalized system of mathematics, rich enough to contain recursive arithmetic, necessarily cotains undecidable propositions, that is propositions which can neither be proved nor disproved, by the proof resources of the system and further deduced that a consistent system cannot prove its consistency by means of the proof process alone which the system admits.

But it seems to be un-natural to confine to intrinsic proof only. An extrinsic proof of consistency, that is, a consistency proof by methods which transcend the limits of the system may sometimes be available.

An extrinsic proof of the consistency of the arithmetic of natural numbers was suggested by Gerhand Centzen[9].

He utilised transfinite induction (over the ordinals of the second class) to proof that certain reduction process which his proof employs terminate in a finite numbers of steps.

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(7) Hilbert and Bernays,—Grundlagen der Mathematik P. 357

(8) Gödel, "Über formal unentscheldbare satze, Monatshfte for Mateematic and Physik 1931."

(9) G. Gentzen, Die widers pruchs-rfeiheit der reinen zahlen theorie, Mathematische Annalen 112, (1936) pp. 493-535.

The work of Gentzen represents the most important attempt to establish the consistency of arithmetic. He has been able to prove the consistency of the whole of arithmetic on the basis of a part of arithmetic which does not include the law of the excluded middle and certain transfinite methods.

But Gentzen's extrinsic proof of consistency has of course the defect—in relation to an intrinsic proof—that the system in which the proof is carried out is not the system whose consistency is proved, so that our confidence in the freedom from contradiction of arithmetic remains based on a belief in the freedom from contradiction of some wider system.

Decision Problem : Next we come to the desision problem for the arithmetic of natural numbers. The decision problem for a system is the problem of devising a mechanical test for the truth or falsity of all its propositions, in a finite number of steps. If such a method exists, it may be called, a decision method or decision procedure of the system.

In a certain elementary system concerning the arithmetic of natural numbers, the decision problem has been solved affirmatively by Presburger,[10] containing no insoluble problems. The system of Presburger is obtained by embodying the axioms of propositional culculus and the axioms of identity with the axioms of arithmetic and all these arithmetical axioms are well defined in terms of a single operation only, in which the sign of multiplication never occurs.

Throughout the system, we shall suppose that the variables p, q, r......range over the class of propositions, while the veriables x, y, z......a, b, c......range over the class natural numbers.

(i) Axioms of Implication.

1. $p \longrightarrow r \longrightarrow q \longrightarrow r. \longrightarrow .p \longrightarrow q.$

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(10) Presburger, Uber Volstandigkeit cines gewissen system der Arithmetic", Comptes rendus du congress des Pays Slaves Warsaw (1929).

2. $\neg p \longrightarrow p . \longrightarrow . p.$

3. $\neg p . \rightarrow : p \longrightarrow q$

(ii) Axioms of Indentity.

4. $a = a.$

5. $a = c \longrightarrow b = c . \longrightarrow . a = b.$

(iii) Axioms of Arithmetic.

6. $a = b \longrightarrow a + c = b + c.$

7. $a + c = b + c \longrightarrow . a = b.$

8. $a + b = b + a.$

9. $a + (b + c) = (a + b) + c.$

10. $a + o = a$

11. $\exists b[a + b = c]$

12. $a + a = b + b \rightarrow a = b.$

$a + a + a = b + b + b . \rightarrow . a = b.$

$\underset{\alpha}{\underline{a + a + a \ldots\ldots}} = \underset{\beta}{\underline{b + \ + b \ldots + b}} . \longrightarrow . a = b.$

which may be written in the form,

$\alpha a = \alpha b . \rightarrow . a = b.$

13. $\exists x[x + x = a \vee x + x + 1 = a \vee x + x + x + 1 + 1 = a$

$\ldots\ldots\ldots\ldots \ldots\ldots\ldots . \ldots].$

14. $\neg(a + a + 1 = 0]$

$\neg[a + a + a + 1 = 0]$

$\ldots \quad \ldots\ldots\ldots$

$\neg[\alpha a + 1 = 0].$

This elementary system of fourteen axioms can be embedded syllogistically in a richer system P containing denumerable propositions of the form $\exists x[\alpha x + a = b]$ which can easily be transformed into the equivalent form $a \underset{\alpha}{\equiv} b$ † and if one belongs to P, the other will also belong to P.

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‡ $a \underset{\alpha}{\equiv} b$ means $a \equiv b \pmod{a}$

We shall now consider the decision problem in the system P. The propositions in P are either primitive (simple) or compound. As regards the decidability of primitive propositions the following observation may be noted. Primitive propositions are without free* variables. Further they can be reduced either to the form $a=b$ or $a \underset{\alpha}{\equiv} b$ or one in the negative form. Primitive propositions can easily be transformed into one of the following forms :—

$0=1$, $1+1\ldots\ldots+1=0$, $0\underset{\alpha}{\equiv}0$, $\underbrace{1+1\ldots\ldots+1}_{\beta}\equiv 0$ or one in the negated form. Every such proposition is decidable. For example, $\underbrace{1+1+\ldots\ldots1}_{B}\underset{\alpha}{\equiv}0$ is equivalent to the proposition, 'β is divisible by α', while $\neg(\underbrace{1+1\ldots\ldots+1}_{\beta}\underset{\alpha}{\equiv}0)$ is equivalent to β is not divisible by α.

As regards the decidability of componnd propositions, the following Lemmas are added from the functional calculus.

Lemma 5. A propositicn of the form $\exists x\,[r_1 \cdot r_2 \ldots\ldots r_n]$ where r_1 is a negated or un-negated primitive proposition can be transformed into an equivalent form $[r'_1 \cdot r'_2 \ldots\ r'_n]$, where r' i is a negated or un-negated primitive preposition and in which no variable occurs.

Lemma 6. Every proposition without free variables can be transformed into a disjunctive normal form.

Lemma 7. Every proposition without free variables is decidable. For by Lemma 6, such proposition can be transformed into a conjunctive or disjunctive normal form and every such normal form is decidable.

Here we shall confine ourselves to the decidability of compound propositions containing only two primitive propositions but the method can be extended for the decidability of propositions containing n primitive propositions.

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*Universal and existential quantifiers are only bound variables while the rest are free variables.

Compound propositions are without free variables and further their primitive propositions are of the form $a=b$ or $a\underset{\alpha}{\equiv}b$ or $a\neq b$ or $\neg(a\underset{\alpha}{\equiv}b)$. But $\neg(a\underset{\alpha}{\equiv}b)$ is reduced to the following compound proposition.

$$a+1\underset{\alpha}{\equiv}b \vee a+1+1\underset{\alpha}{\equiv}b \vee \frac{a+1+1\cdots\cdots}{\alpha-1\ (\text{times})}+1\underset{\alpha}{\equiv}b.$$

Therefore, we shall consider compound proposition constructed from the three relations only $=, \neq$ & $\equiv$. Here we shall have only following six kinds of propositions.

I. Equation and Equation. II. Congruence and Congruence. III. Non equation and Non equation. IV. Equation and Non equation. V. Equation and Congruence. VI. Congruence and Non equation.

We shall presently see that the Lemma 7 holds good for all the six kinds of propositions. Here we shall sketch the proof for the first kind of propositions only and the proof for the rest will follow from similar arguments.

1. $\exists x[\alpha x+a=b.\ \alpha' x+\alpha'=b]$, if β is the least common multiple of α and α', the proposition reduces to $\exists x[\beta\, x+c=\mathbf{d}.\beta x+c'=\mathbf{d}']$.

In other words $(c\underset{\beta}{\equiv}d).\ (d+c'=d'+c)$.

II. $\exists_x[\alpha x+a\underset{\beta}{\equiv}b.\quad \alpha' x+a'\underset{\beta'}{\equiv}b']$.

The Lama can also be proved to hold good for this proposition.

III. $\exists_x[\neg(\alpha x+a=b).\ \neg(\alpha' x+a'=b')$. The proposition reduces to $0=0$.

IV. $\exists_x[\alpha x+a=b.\neg(\alpha' x+a'=b')]$.

This reduces to $(c\underset{\beta}{\equiv}d).\ \neg(d+c'=d'+c)$.

where β is the least common multiple like I.

V. $\exists_x[\alpha x+a=b.\neg(\alpha' x+a'\underset{\beta}{\equiv}b')]$.

VI. $\exists_x[\alpha x+a\underset{\beta}{\equiv}b.\neg(\alpha' x+a'=b')]$.

The Lemma can also be proved to hold for V. and VI.

In P, all the propositions with only one sign of + occur and we have seen that it contains no insoluble problems. But if we introduce the multiplication sign as well, the decision problem will not be affirmative. For in 1936, Alonzo Church[11] (on the basis of the work of Gödel)$_a$ for the first time, established that the arithmetic of natural numbers with + and · is undecidable.

Applying Church's result to a complete set of axioms for mathematics, we can conclude that there can never be a systematic or mechanical procedure for solving all mathematical problems. In other words, mathematicians will never be replaced by a machine.

Thus we conclude that the arithmetic of natural numbers, when viewed from an angle of decision prblem, assumes an aspect of incompleteness.

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(11) Alonzo Charch, "An unsolvable problem of number theory" Am. Jour. Math. 1936.

APPENDIX 2

Paradox, Decision Problems and Algebraic Logic

Here we shall consider briefly logical paradox. Decision problems for F O F C and Algebraic Logic

I. Paradox

(A) Semantic paradox

(i) The Liar Paradox. A man says "I am lying". If he is lying, then what he says is true and so he is not lying. If he is not lying then what he says is true, and so he is lying. Thus in both cases he is lying as well as not lying.

(ii) The Richard paradox. (1905), which is connected in the work by H. Lebesgue on analytically representable functions.

(iii) The Berry paradox (1906), which is a simplified form of the Richard Paradox.

(iv) The Grelling paradox (1908).

(v) The paradox of the Barber. The following regulation has been promulgated by the Council of a village.

"Any male inhabitant, who does not shave himself is compelled to be shaved by the village barber and at the sametime, the barber is strictly forbidden to shave any body who is in the habit of shaving himself.

Cleary the village barber will be put in a painful dilemma. It he does not shave himself, he will be obliged to do so; if he forms the habit of shaving himself, this will be strictly forbidden to him.

(B) Logical paradox

(i) The Russel paradox (1903).

It is the paradox of the class $\{x|x \overline{\in} x\}$

i. e. The Class of those Classes that are not members of themselves.

Now consider the formula

$$x \overline{\in} x$$

According to the principle of abstraction this formula determines a set R such that $x \in R$ iff $(x \overline{\in} x)$.

In particular,

$$R \in R \text{ iff } (R \overline{\in} R);$$

which is logically equivalent to the contradiction

$$R \in R \text{ and } (R \overline{\in} R).$$

The Russell paradox is the prototype of a whole family of paradoxes. For instance

$$(x) \ \{x \overline{\in} x\},$$
$$(y) \ \{y \overline{\in} x \cdot x \overline{\in} y\},$$
$$(y)(z) \ \{y \overline{\in} x \,.\, z \overline{\in} y.\ x \overline{\in} z\}$$

(ii) The cantor paradox (1899). The cardinal number of a set x is less than the cardinal number of the set P (X) of subsets of X. In particular, the cardinal number of the set U of all sets is less than the cardinal number of P (U), although P (U) $\subset$ U.

(iii) The Burali—Forti's Paradox (1897) .

The ordinal numbers, taken in their natural order, form a well-ordered set, and this set accordingly itself has an ordinal number $\sim$. But the ordinal number of any segment of the set of ordinal numbers exceeds every number of that segment, and therefore $\sim$ exceeds any ordinal number whatever.

2. The Decision Problem for formalised theories.

The term "decision problem" which is due to Hilbert, refers to the general problem of setting up, for a given formal system Σ a systematic and constructive process whereby it can be determined whether or not any selected formula of Σ is valid.

The decision problem is solvable for the propositional calculus, but for the first order functional calculus (FOFC) no general solution of it is possible. Church has proved that there can be no decision procedure for this calculus that is constructive in the sense of general recursive.

We shall mention here the existence of particular decision procedures. Many results have been obtained concerning decidability, with respect to identical truth or satisfiabitily, for special classes of formulas of the FOFC. In 'Special Cases of the decision problem' Church gave a survey of existing knowledge in this field; and the subject has been considered fully by Ackermann, in soluable cases of the Decision problem. It was shown earlier by Lo'wenheim that a decision procedure exists for formulae of FOFC in which no functional variable has more than one argument place. Quine has also presented a new decision procedure for monadic schemata which seems convenient enough for practical and pedagogical use. A particular decision problem that has been considered is the 'word problem' for groups and other related mathematical systems. A new proof of the unsolvability of the word problem for groups has been given by Britton in "The word problems for groups'"

The decision problem for an elementary arithmetic of natural number has already been considered in appendix I.

3. Algebraic Logic. Algebraic Logic starts from certain special logical considerations, abstracts from them,: places them into a general algebraic context and generalisation makes contact with topology and functional analysis. It is a part of pure mathematics in which the concepts that constitute the skeleton of modern symbolic logic can be discussed in algebraic language.

3. I Boolean algebra. The father of algebraic Logic is George Boole. The shortest definition of Boolean algebra is in terms of the theory of rings.

A Boolean algebra B is a ring with unit in which each element is idempotent. For example the field of integers modulo 2 is a Boolean algebra. A Boolean algebra has characterisic 2 i. e. $p+p=0$. A natural order relation in B is defined by writing $p \leqslant q$ iff $pq=p$.

The algebraic structure and the order structure are compatible. For the algebraic zero is also the order zero, i. e, (the least element of the algebra), and the algebraic unit is also the order unit (i.e, the greatest element of the algebra). In other words $0 \leqslant b \leqslant 1$ for each p in B. With respect to the order, the algebra turns out to be complemented modular lattice. The lattice operations can be expressed in terms of the given algebraic operations. For the complement p' of p is $1+p$;

the infimimum of p and q denoted by $p \wedge q$ is $p \cdot q$; and the supremum $p \vee q$ is $p+q+pq$.

A Boolean algebra can be defined in various ways. It can be defined in terms of its partial order (i. e, a distributive and complement lattice with unit and zero is a Boolean algebra) or in terms of complements and suprema, or in terms of comlements and infima and so on'). Since sum and product can be expressed in terms of complements and infima (ie $pq = p \wedge q$ and $p+q = (p' \wedge q')' \wedge (p \wedge q)'$) it follows that a set B with a unary$'$ and a binary operation $\wedge$ satisfying the following appropriate conditions is a Boolean algebra.

1. $\wedge$ is associative and commutative; 2. $p \wedge q' = r \wedge r'$ iff $p \wedge q = p$

Filters and ideals in a Boolean algebra are next introduced. A nonempty subset I of a Boolean algebra B is an ideal if it contains $p \vee q$ whenever it contains both p and q and if it contains $p \wedge q$ whenever it contains p. A Boolean algebra is a ring and a Boolean ideal is, therefore, an ordinary algebraic ideal. A nonempty subset F of B is called a filter if it contains $p \wedge q$ whenever it contains both p and q and if it contains $p \vee q$ whenever it contains p for all q in B. A necessary and sufficient condition that a subset F of a Boolean algebra be a filter is that $1 \in F$ and that if $p \in F$ and $(p \rightarrow q) \in F$, then $q \in F$ (Modues ponens). If the elements of a Boolean algebra are propositions of some theory, then the elements of the filter F are thought of as provable propositions. Each of the two concepts filter and ideal is in a sense the Boolean dual of the other. For, if F is a filter in a Boolean algebra B and if I be the set of all those elements p of B for which $p' \in F$, then I is an ideal in B; the procedure of making a filter out of an ideal works similarly. Just as filters arise in the theory of provability, ideals arise in the theory of refutability (where a proposition is refutable if its negation p' is provable). Logic is usually studied from the '1' approach, i.e, emphasis is on the truth and provability and consequently on filters. we shall next consider logic from the dual '0' approach using algebrical concept of ideal.

3.2 **Boolean logics.** A Boolean logic is an ordered pair <B, I>, where B is a Boolean algebra and I is a Boolean ideal in B. The elements of B will be called propositions, the elements of I will be called refutable propositions. A Boolean logic is called consistent if for no proposition p is B are both p and p' provable or, equivalently, if for no p in B do both p and p' belong to I. As I is an ideal, <B,I> is consistent iff the ideal I is proper or roughly speaking the set of propositions that are refutable in B is not too large. Inconsistent logics, i. e, logics of the form. < B, B> are not interesting from either the algebraic or the logical point of view. For a consistent logic <B, I> the concept of completeness has an algebraic formulation : the logic is complete iff I is the maximal proper ideal in B. A necessary and sufficient condition that a Boolean logic <B, I> be both consistent and complete is that $B/I = \{0,1\}$, where B/ I is the quotient Boolean algebra. Note that if a proposition in B is refutable then its image in B/I (its equivalence class modulo I) is equal to 0 and if a proposition in B is provable, then its image in B / I is equal to 1.

The existential quantifier is a mapping $\exists$ (Σ) of a Boolean algebra (propositional function) into a certain sub-algebra (constant functions) satisfying the following conditions :—

(1) $\exists\, O=0$;

(ii) $p \leqslant \exists p$;

(iii) $\exists(p \vee q) = \exists p \vee \exists q$;

(iv) $\exists\exists p = \exists p$;

(v) $\exists(\exists p)' = (\exists p)'$

It is interesting to note that $\exists$ with the conditions (i) to (iv) is an analogue of closure operator on a topological space.

Dually,a universal quantifier is a mapping $\forall$ (II) of a Boolean algebra lnto itself satisfying the conditions :

(i)′ $\forall'1=1$;

(ii)′ $\forall p=p$;

(iii)′ $\forall(p\wedge q)=\forall p\lambda\forall q$;

(iv)′ $\forall\forall p=\forall p$;

(v)′ $\forall(\forall p)'=(Ap)'$

It can easily be verified that $\forall$ bears the same relation to the intuitive 'all' as $\exists$ bears to "some".

If $\exists$ is an existential quantifier on a Boolean algebra B and if a mapping $\forall$ of B into self is defined by $\forall p=(\exists p')'$ then $\forall$ is a universal quantifier.

3. Monadic algebra. A monadic algebra is a pair $\langle A, \epsilon\rangle$, where A is a Boolean algebra and ϵ is a quantifier on A. The word "mondic" serves as a reminder of the one additional operation that distinguishes monadic algebras from Boolean algebras The elementary theory of mondic algebra includes monadic subalgebra monadic homomorphism, ideals.

Free monadic algebras can be defined by the techniques used in the study of Free Boolean algebras, the analogue of the propositional calculus is called the monadic functional calculus.

34 **Monadic Logic** : The concept of monadic Logic arises in connections with tne theory of provability and refutability in the monadic functional Calculus. A monadic logic is a pair $\angle$ M, I $>$, where M is a monadic algebra (with quantifier $\exists$) and I is a monadic ideal in M, which is also denoted by $\angle$ $\angle$M, $\exists>$, I$>$. The fact that a monadic ideal is a Boolean ideal invariant under the application of $\exists$ corresponds to the logical fact that if p is a refutable propositional function, then $\exists p$ is also refutable. An element p of a monadic algebra is called closed if $\exists p=p$. Intui

tively closed propositions correspond to Constant functions i-e propositions. There is a natural way of associating a Boolean logic $\angle B_0 I_0 >$ with each monadic logic $\angle B, I>$, where the algebra B_0 is the set of all Closed elements of B and the ideal J_0 is the intersection of I with B_0. A monadic algebra $< B, I >$ is called syntatically Consistent (or complete) if the associated Boolean logic $\angle B_0, I_0 \urcorner$ is consistent (or complete) A quantifier algebra is a triple $\angle B, I, \ni>$, where B is a Boolean algebra, I is a set, and $\ni$ is a function from sub set of I to quantifiers on B such that

$$\ni(\phi)p=p \text{ whenever } p \in A, \text{ and}$$
$$\ni(J)\ \ni(K) = \ni(JUK), \text{ whenever}$$

J and K are sub sets of I. A quantifier algebra $\angle A, I, \in>$ for which the set I of variables consists of exactly one element may be identified with the monadic algebra $\angle A, \ni <I> >$. The degree of a quantifier algebra is defined to be the cardinal number of the set of its variables and that the quantifier algebras of degree are simply monadic algebras.

The quantifier algebra $< B, I, \ni >$ is locally finite if to each element p of B, there corresponds a finite sub set J of

$$I \text{ such that } \ni(I-J)p=p$$

i. e. no propositional function will depend on infinitely many variables.

3.5 Cylindric Algebra : A Cylindric algebra is a quantifier algebra together with certain distinguished elements, the distinguished elements play the role of sentences that assert equations among variables. The theory of Cylindrical algebra is an algebraic tool for studying Calculi with equality.

Polyadic Boolean algebras are related to although not identical with the Cylindrical algebras.

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